The Case Against Dark Matter and Modified Gravity

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Outline

Math: 2^{nd} Order Differential Equations

- The importance of trivial solutions
- Oscillatory vs. non-oscillatory solutions
- Examples:
  - Parabolic Cylinder Equations
  - Inhomogeneous Bessel n=0 Equation
  - Classical Lane-Emden Equations

Physics: Rotating Self-Gravitating Fluids

- Lane-Emden equations with rotation
- The case in favor of Newtonian Dynamics

Common Misconceptions

- Trivial solutions are of no importance, hence the name.
- In the Cauchy (IVP) Problem
  - boundary/initial conditions fully determine the solutions.
  - without them the differential equation does not describe physics.

As a result:

- No-one is “asking” which solutions the equation itself favors.
- Singular solutions appear to be unrelated to IVP solutions and they are usually thought to be unphysical.

We shall see that 2^{nd} order ODEs have no regard for the IVP and prefer instead their own intrinsic (trivial) solutions. Not only that, but they also force the IVP solutions to conform.

Trivial Solutions

- $a(x)y'' + b(x)y' + c(x)y = 0$
- $y = 0$ is a trivial solution.
- $y=0$: Intrinsic to the ODE.
  The ODE does not care about boundary conditions.
  So $y=0$ is stronger (more preferable) than the Cauchy Problem!
- $y=0$: Attractor of oscillatory IVP solutions.
- Method for finding such solutions: Set the y term equal to RHS, and separately the derivative terms equal to 0.
  (Why does it work though? Let's talk timescales here…)

More Misconceptions

Back to math:

- $a(x)y'' + b(x)y' + c(x)y = 0$

By analogy to the Harmonic Oscillator (a,b,c = const.):

- The term $b(x)y'$ represents damping - resists oscillations
  - Incorrect, $b=1/x$ is an inertial term and represents no damping.

- A $c(x) > 0$ with $b^2 - 4ac > 0$ usually leads to oscillatory behavior
  - Incorrect, this only works for constant a,b,c and for some special (b=0) cases.

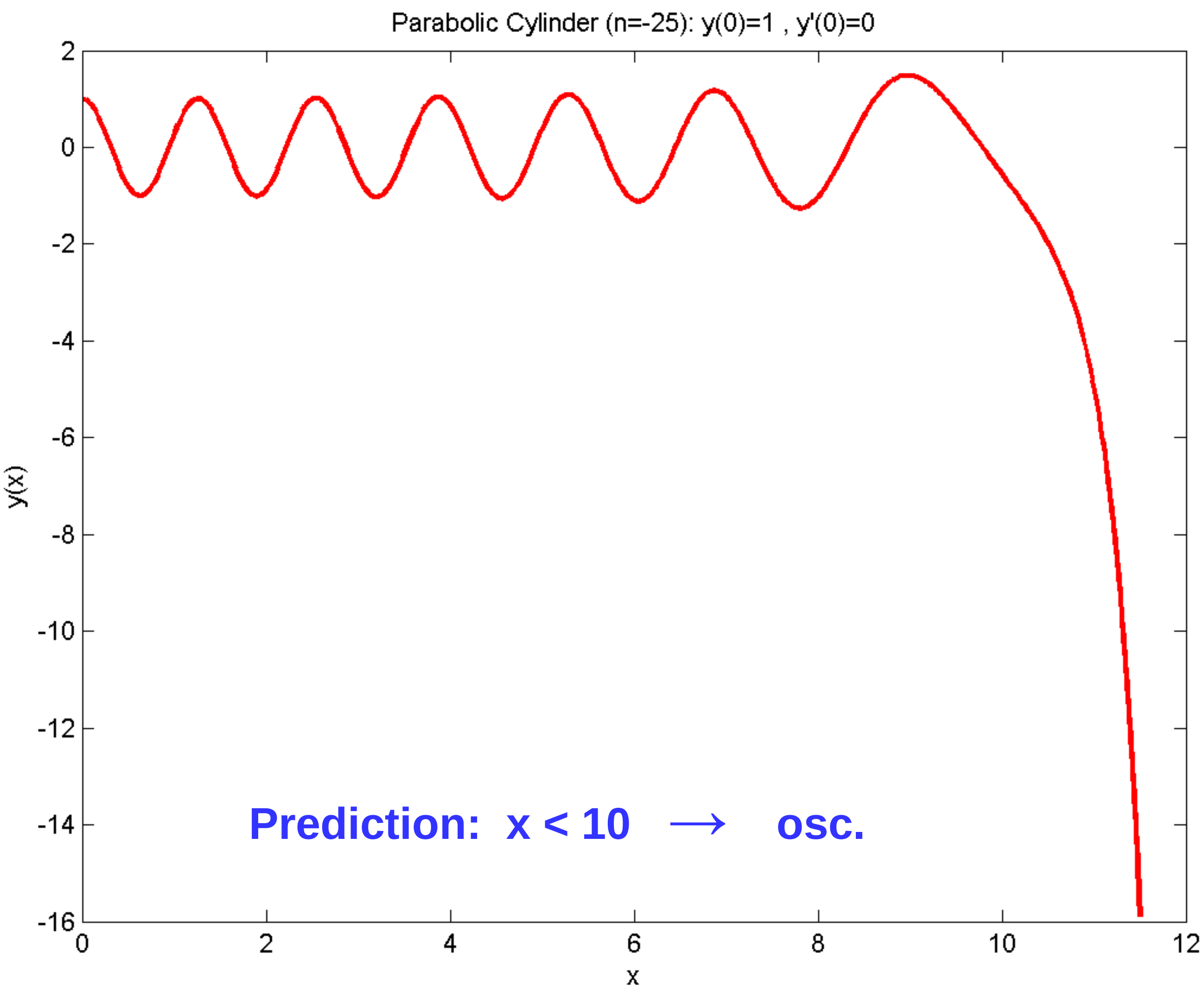
For details, please see a series of papers posted in ResearchGate

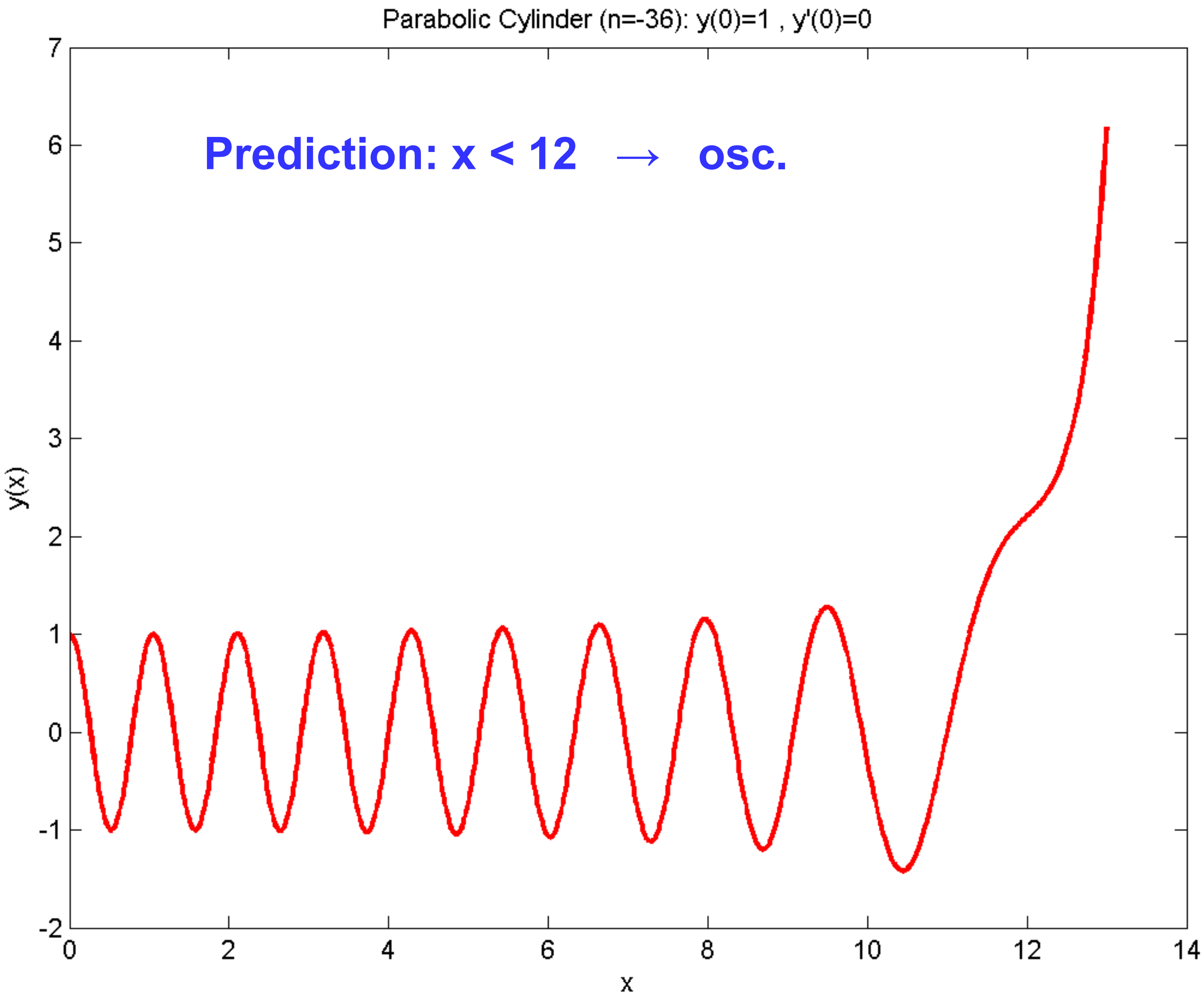
Example 1

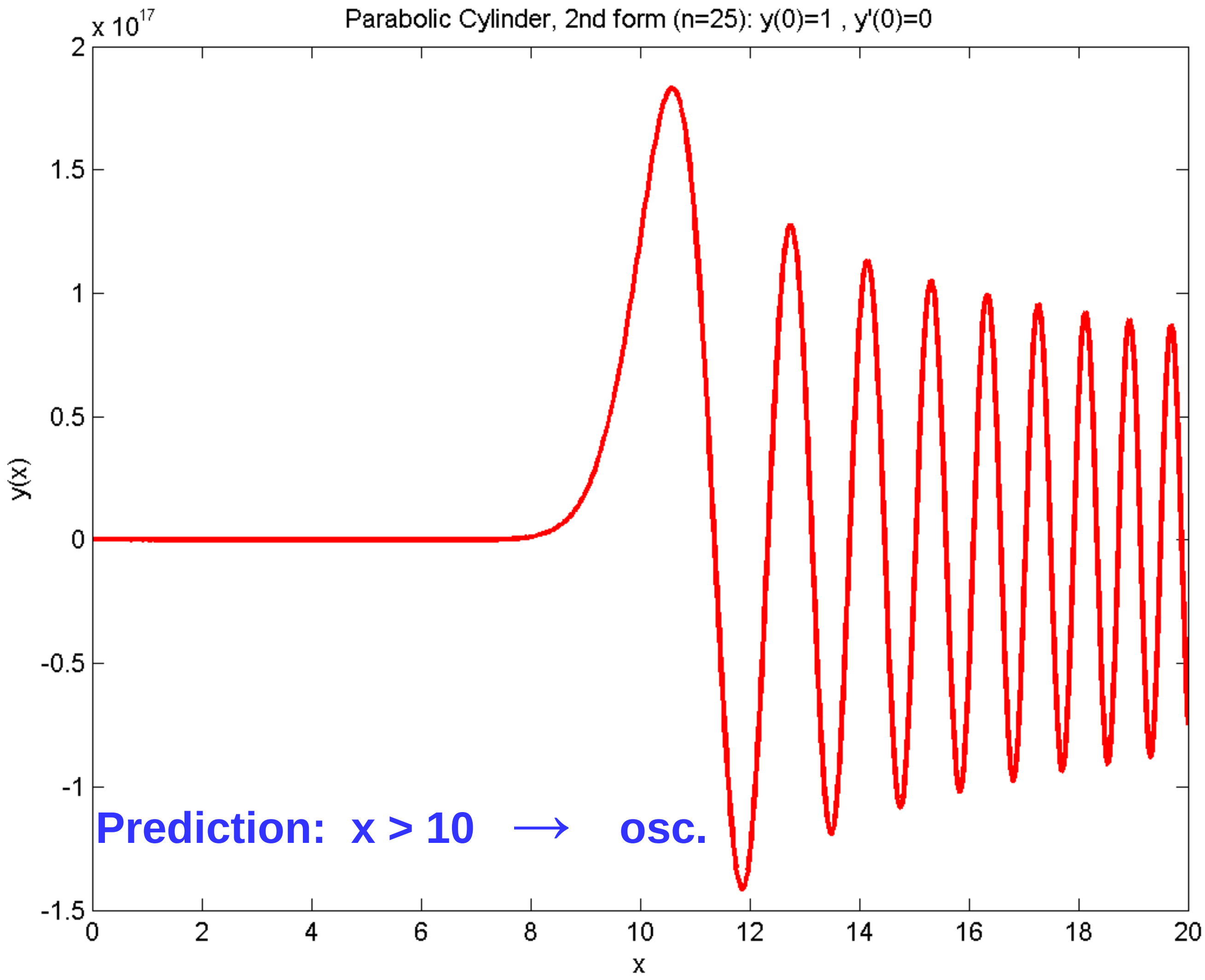
- Parabolic Cylinder

$$y'' - (x^2/4 + n)y = 0$$

$$y'' + (x^2/4 - n)y = 0 \qquad 2^{nd} \text{ form}$$


Parabolic Cylinder (n=-25): y(0)=1 , y'(0)=0
y(x)
x
2
0
-2
-4
-6
-8
-10
-12
-14
-16
0
2
4
6
8
10
12
Prediction: x < 10 ⟶ osc.


Parabolic Cylinder (n=-36): y(0)=1 , y'(0)=0
Prediction: x < 12 → osc.
7
6
5
4
3
2
1
0
-1
-2
y(x)
0
2
4
6
8
10
12
14
x

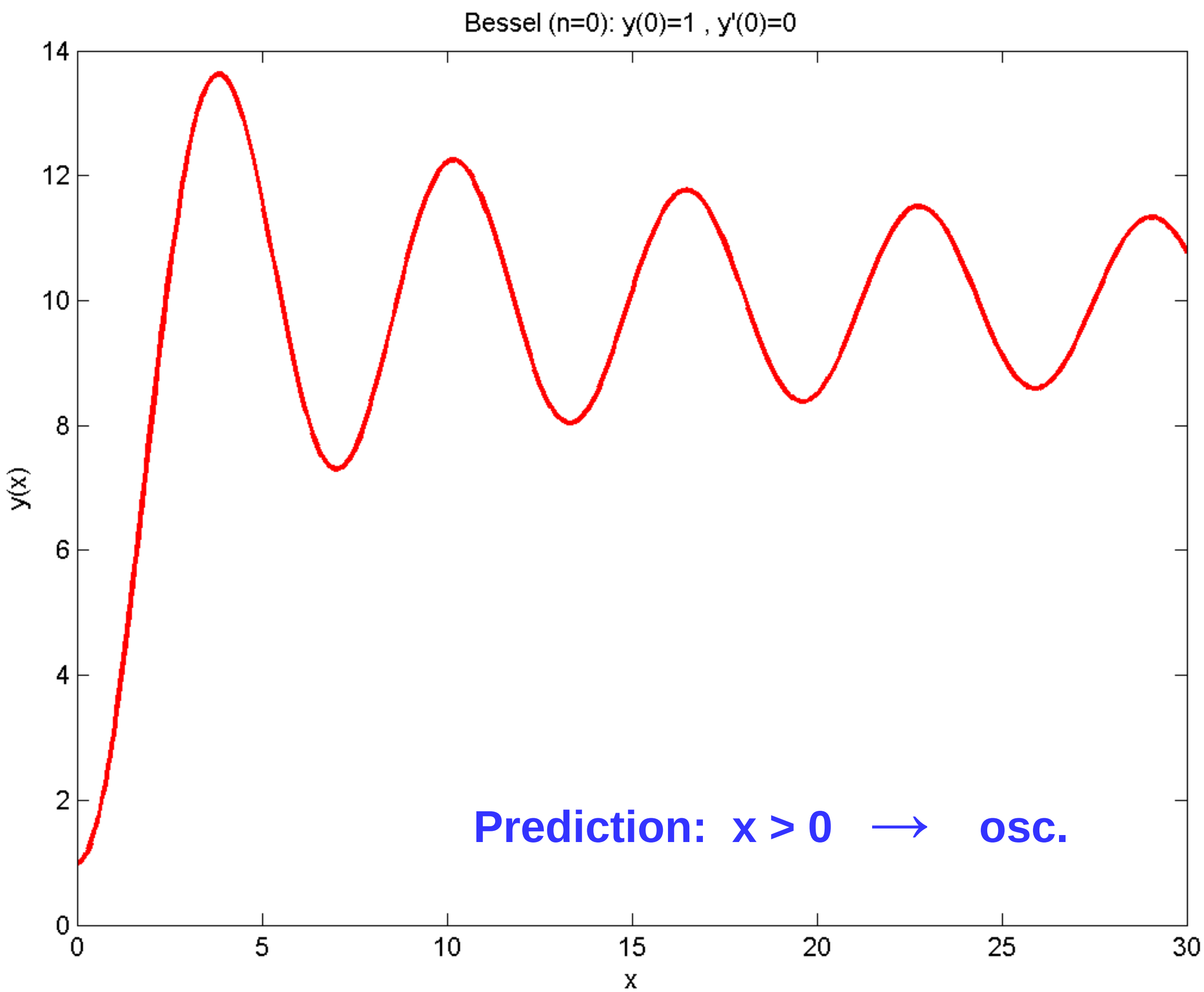

Parabolic Cylinder, 2nd form (n=25): y(0)=1 , y'(0)=0
x 10^17
2
1.5
1
0.5
0
-0.5
-1
-1.5
y(x)
0
2
4
6
8
10
12
14
16
18
20
x
Prediction: x > 10 ⟶ osc.

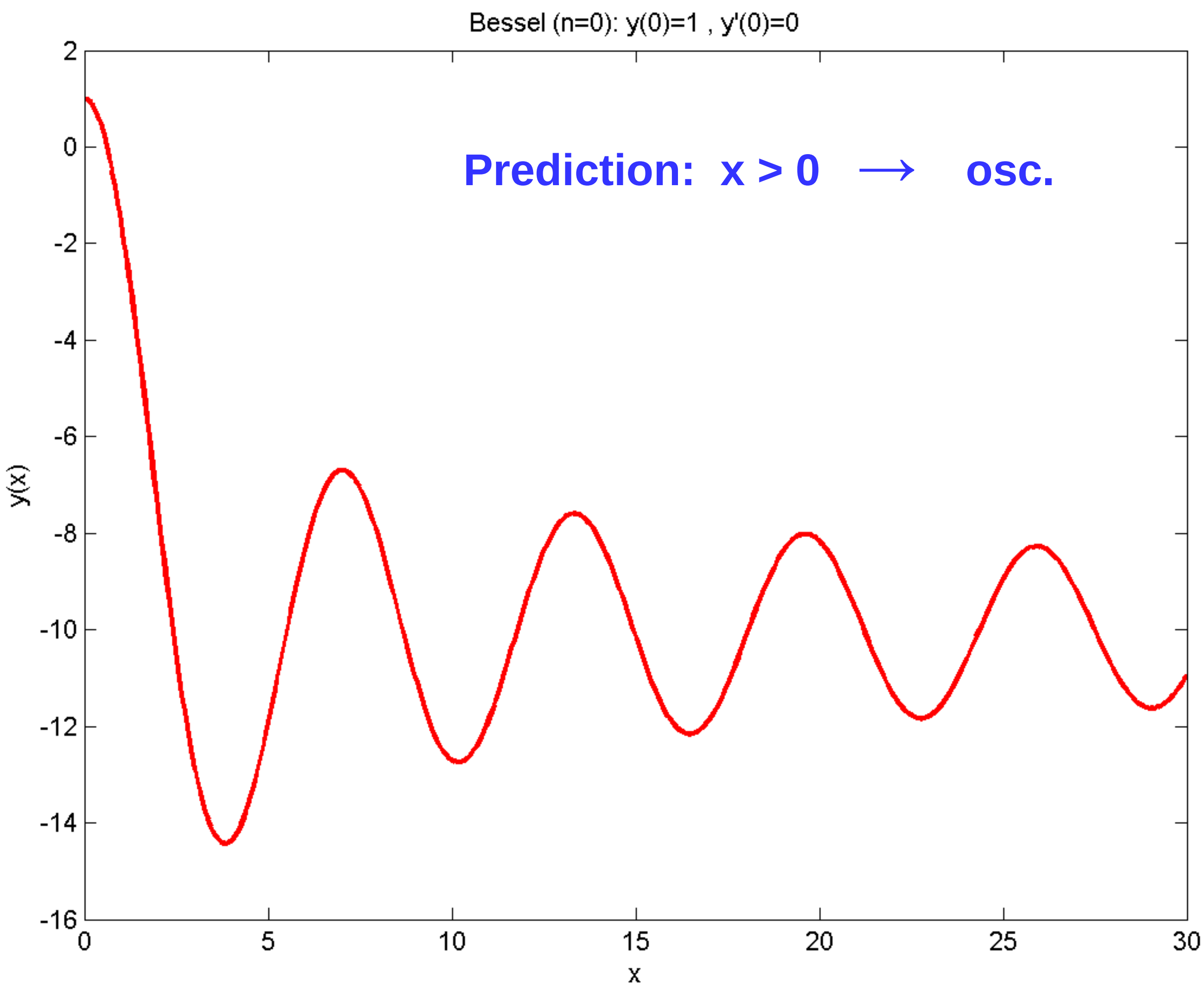
Example 2

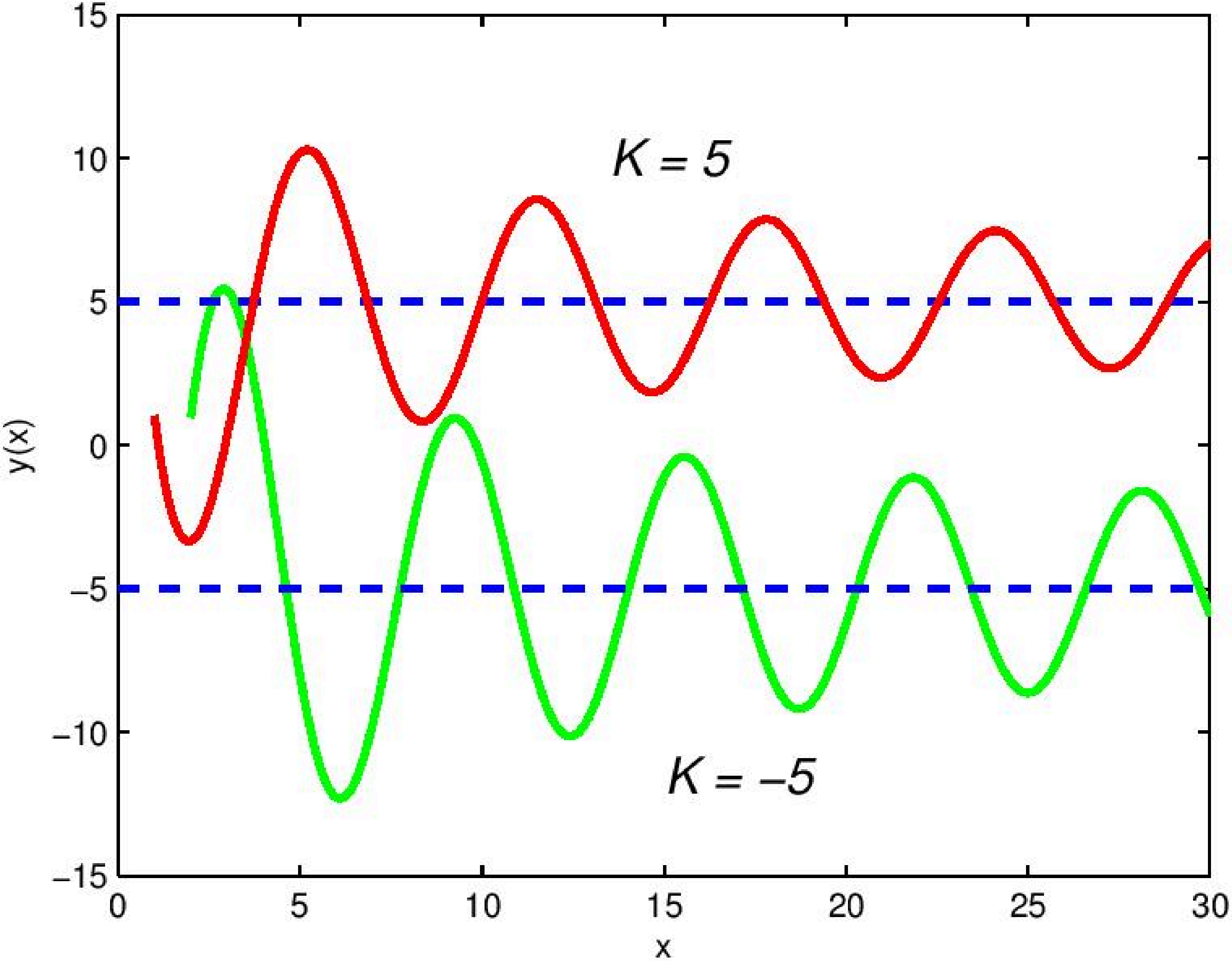
- Inhomogeneous Bessel of zero order

$$x^2y'' + x \cdot y' + x^2y = K$$

where K = -10, -5, +5, or +10.


Bessel (n=0): y(0)=1 , y'(0)=0
14
12
10
8
6
4
2
0
y(x)
0
5
10
15
20
25
30
x
Prediction: x > 0 ⟶ osc.


Bessel (n=0): y(0)=1 , y'(0)=0
Prediction: x > 0 ⟶ osc.
y(x)
x
2
0
-2
-4
-6
-8
-10
-12
-14
-16
0
5
10
15
20
25
30



15
10
5
0
−5
−10
−15
y(x)
0
5
10
15
20
25
30
x
K = 5
K = −5

Example 3

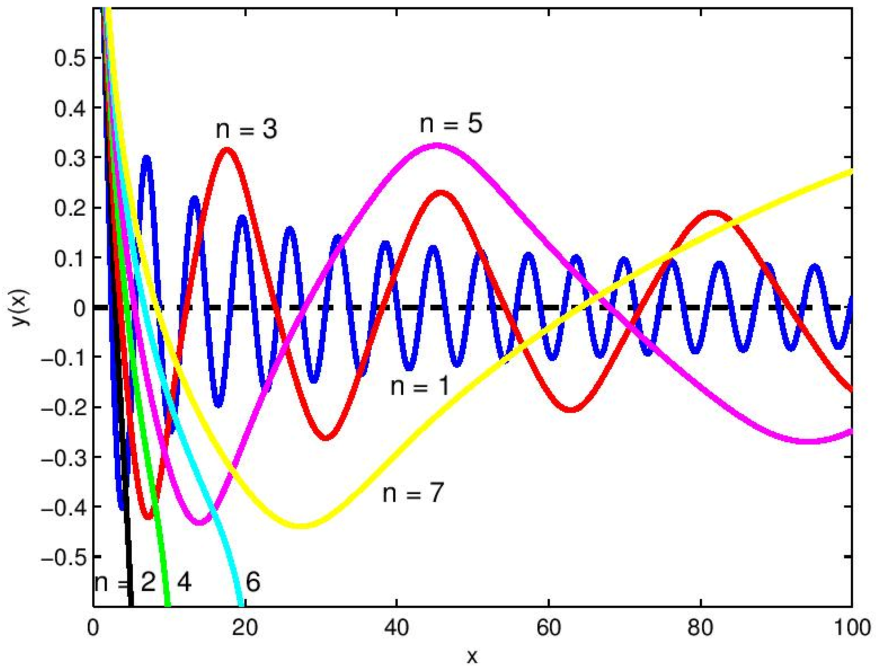
- Polytropic Lane-Emden Equations (n = positive integer)

$$y'' + y'/x + y^n = 0$$ cylindrical form

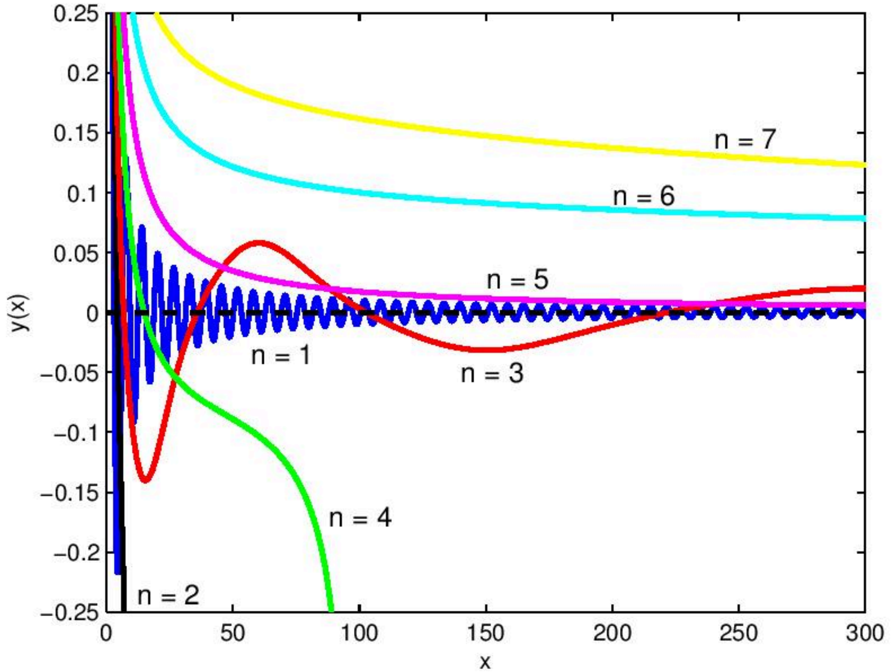
Criterion: n=odd int. → osc.

$$y'' + 2y'/x + y^n = 0$$ spherical form

Criterion: n=1, 3 only → osc.



Cylindrical Form, 1/x coeff.
n=odd int. → osc.



Spherical Form, 2/x coeff.
n=1, 3 only → osc.

Cylindrical Lane-Emden with Rotation

- Isothermal $$c_o^2 \cdot \frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\ln\tau + \tau = \frac{1}{x}\frac{dv^2}{dx}$$
- Polytropic $$nc_o^2 \cdot \frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\tau^{1/n} + \tau = \frac{1}{x}\frac{dv^2}{dx}$$
- Intrinsic Solution $$\tau(x) = \frac{1}{x}\frac{dv^2(x)}{dx}$$

$$\frac{d}{dx}x\frac{d}{dx}(\ldots) = 0$$

Intrinsic Solution

- Isothermal

$$\tau(x) = Ax^{k-1}, \quad (A, k = \text{const.})$$

$$v(x) = \sqrt{Ag(x) + B}, \quad (B = \text{const.}),$$

where

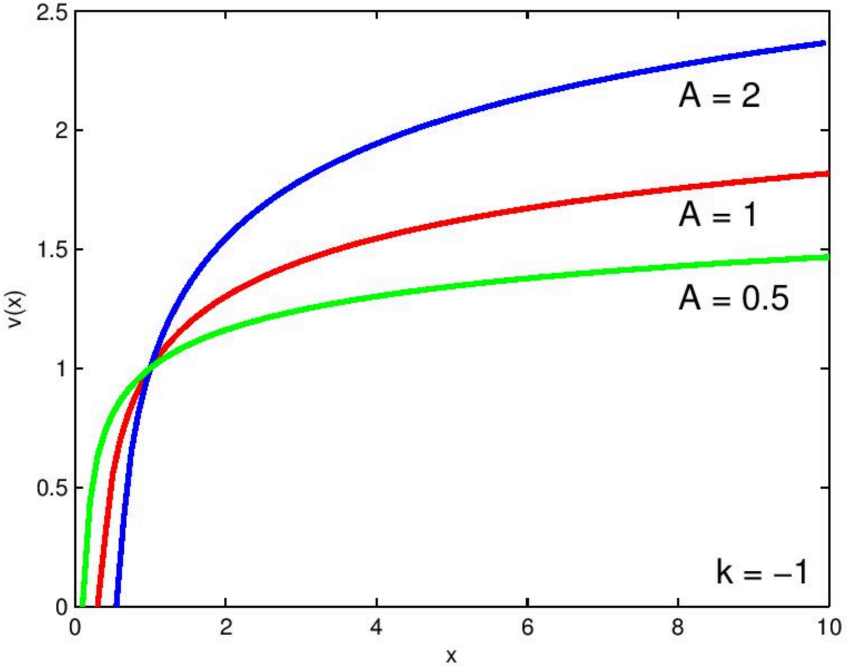
$$g(x) \equiv \begin{cases} x^{k+1}/(k+1)\,, & \text{if} \quad k \neq -1 \\ \ln x\,, & \text{if} \quad k = -1 \end{cases}$$

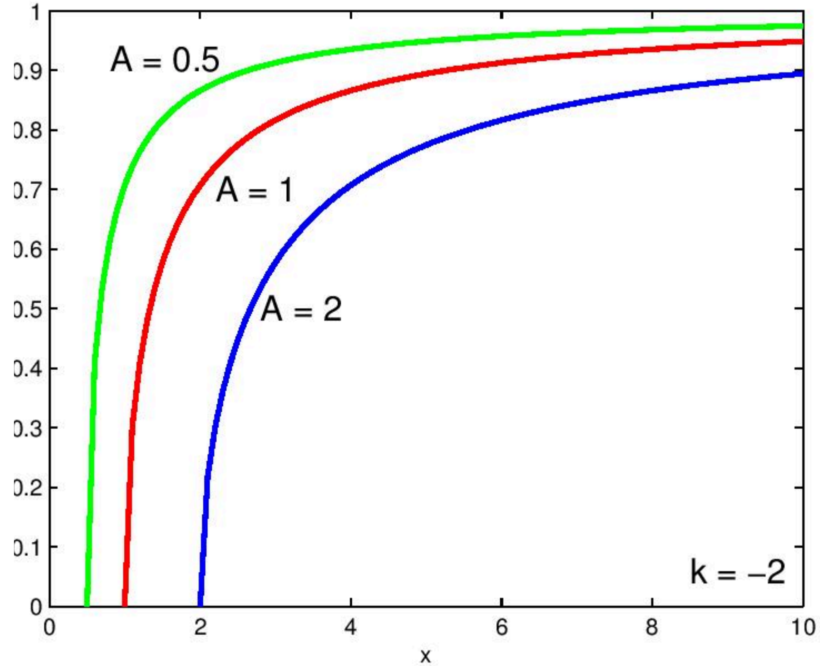
- Polytropic

$$\tau(x) = \left[\ln(Ax^k)\right]^n, \quad (A, k = \text{const.})$$

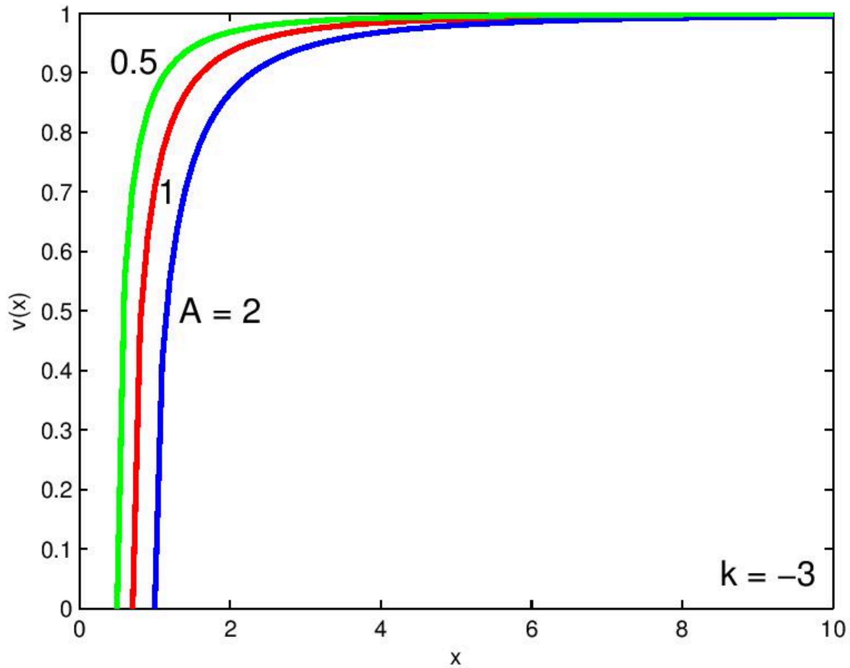
$$v(x) = \sqrt{\frac{A^{-2/k}}{2}\left(\frac{-k}{2}\right)^n \cdot \Gamma\left(n+1, \ln\frac{A^{-2/k}}{x^2}\right) + B}$$

$$A > 0,\ n > 0,\ k < 0,\ Ax^k \geqslant 1\ (\text{i.e., } x \leqslant A^{-1/k})$$

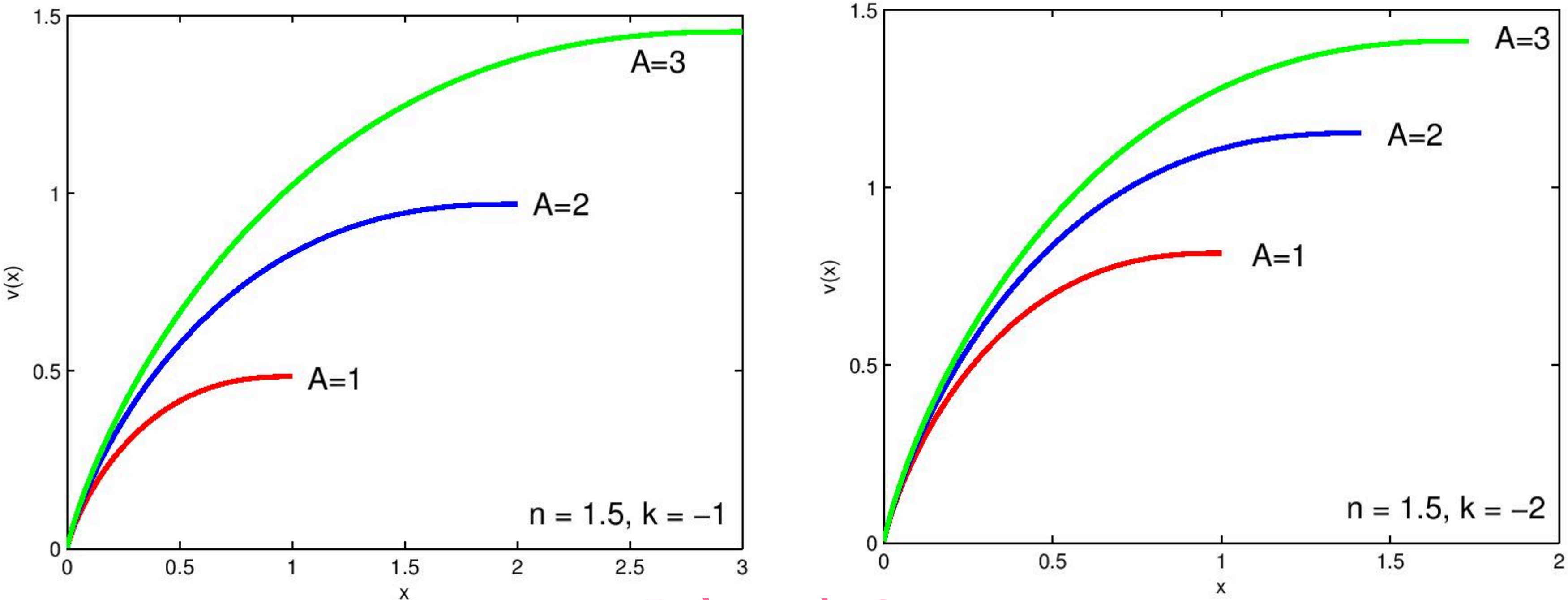


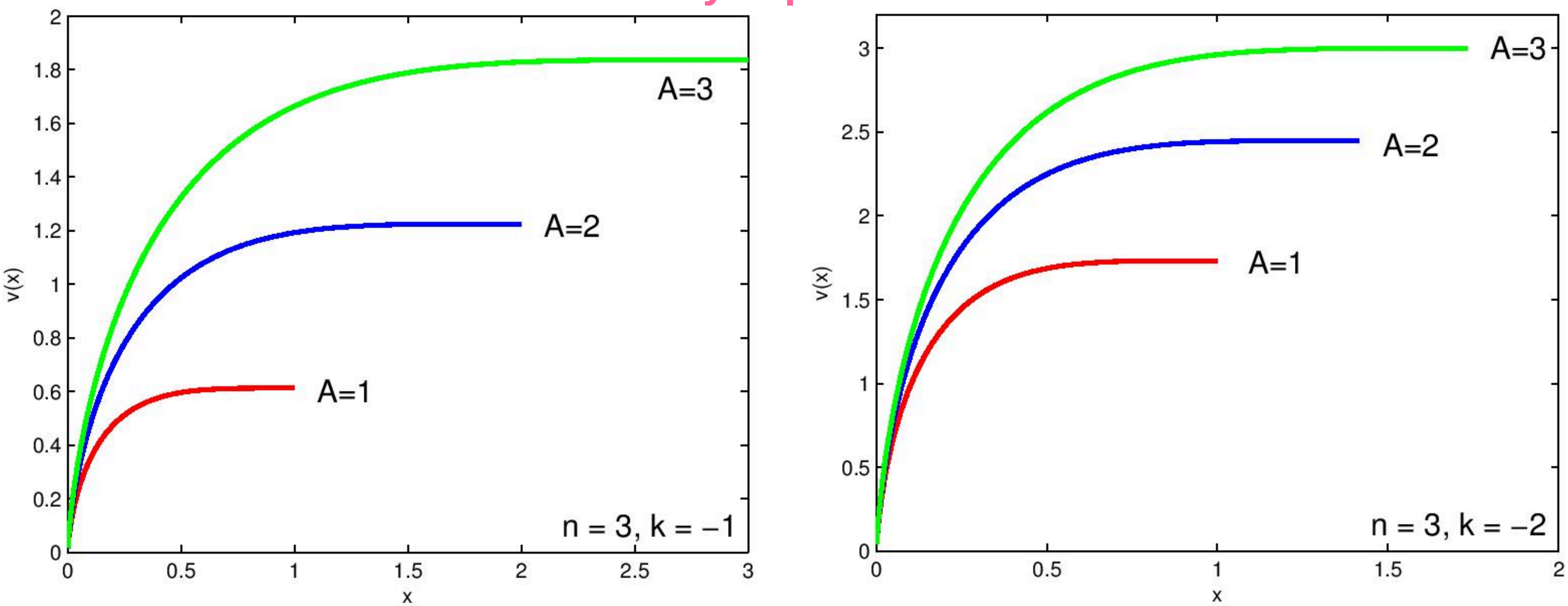


Isothermal Case



Polytropic Case





Physical Interpretation

- Rewrite both Lane-Emden Equations as

$$\frac{1}{x}\frac{d}{dx}\left[\frac{d}{d\ln x}(h+\psi)\right] = \frac{1}{x}\frac{d}{dx}\left[v^2\right]$$

h(x) = specific enthalpy and $\psi(x)$ = Newtonian gravitational potential

- Intrinsic Solution $\frac{1}{x}\frac{d}{dx}\left(\frac{d\psi}{d\ln x}\right) = \frac{1}{x}\frac{d}{dx}\left(v^2\right) \implies \frac{v^2}{x} = \frac{d\psi}{dx}$

$$\frac{d}{dx}\left(\frac{dh}{d\ln x}\right) = 0 \implies \frac{dh}{dx} \propto \frac{1}{x} \implies h(x) \propto \ln x$$

Disks vs. Cylinders

- Enthalpy term ignored for disks: $\left.\frac{d^2 h(x,z)}{dz^2}\right|_{z=0}$
- The nature of pressure to push isotropically: $h(r) \propto \ln r = \frac{1}{2}\ln(x^2+z^2)$
- Radial Laplacian at $z=0$ is indeed zero.

- Combining these equations, we get: $\left.\frac{d^2 h(x,z)}{dz^2}\right|_{z=0} \propto \frac{1}{x^2}$

- where the proportionality constant is $\propto c_o^2$

For more details, please see Appendix in the main paper.