Quantum gravitational anomaly as a dark matter candidate

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Outline

1. Overview, problem statement, and results.
2. Problem of time in canonical quantum gravity.
3. Effective action (EA) in QFT. The background field method.
4. Symmetries of the EA, the Ward identities, and anomalies.
5. QFT on a curved background.
6. Quantum gravitational anomaly. Restoration of general covariance of QFT.
7. Relativistic hydrodynamics in the Fock-Taub representation.
8. Normalization conditions on the EA and the fluid pressure.
9. Quantization of relativistic hydrodynamics in the effective field theory approach.
10. Dark matter (a brief overview).
11. Cold dark matter from the quantum gravitational anomaly. Polytropic equation of state.
12. Summary.

Aim

- The lecture is aimed to acquaint the audience with one of the model of the cold dark matter stemming from the quantum gravitational anomaly of quantum gravity.

The main problems we are going to address and solve in some way

1. Problem of time in canonical quantum gravity (QG). Unitary inequivalence of QFTs corresponding to different choices of the time variable.
2. Problem of the explicit dependence of QFT observables on the choice of vacuum state for quantum fields (it is related to the first problem).
3. Dark matter (DM) phenomenon: at large scales ($\gtrsim 20$ kpc), there are deviations from the GR predictions assuming the known at the present day set of particles.

Overview

Outcomes

1. A dynamical solution to the problem of time in QG (and QFT) is proposed.
2. General covariance (the background independence) of QG is restored at the cost of 4 additional degrees of freedom (the timelike vector field).
3. The dynamics of this field are described by the Euler equations for relativistic perfect isentropic fluid (the "flow of time").
4. In the limit of a weak gravitational field, the state of this fluid is described by a polytropic equation with two universal constants: the polytropic constant and the natural polytropic index. Therefore, this fluid behaves as a cold DM in this regime.
5. If one identifies that fluid with a considerable part of the DM, the value of the polytropic constant can be estimated using the astrophysical data for a local DM distribution. This value agrees with the CDM interpretation of this fluid.
6. The quantum theory of this fluid is constructed using the background field method. So, the complete quantization of gravity and matter fields is achieved, though it results in the non-renormalizable model.

Remark

- The relativistic fluid referred above arises with necessity in QFT with gravity provided the basic principles of QFT (the locality and unitarity) and GR hold and the general covariance (the background independence) remains intact on the quantum level, i.e., the general covariance is a fundamental symmetry of Nature.
- This fluid not only gives a (probable) solution to the DM problem, but also provides a natural solution to the problem of time in QG. Anyway, the latter problem has to be solved, whether unknown dark matter particles exist or not.
- Such a solution to the problem of time was expected long ago (under the name of reference fluid), but only recently was it realized that this solution follows with necessity from the basic principles of QFT. In this way, the equation of state of this "reference" fluid was found.

Problem of time in canonical quantum gravity

There is a huge literature devoted to this problem. I recommend

References:

1. C.J. Isham, *Canonical quantum gravity and the problem of time*, arXiv:gr-qc/9210011.
2. B.S. DeWitt, *The quantization of geometry*, in L. Witten, ed., Gravitation: An Introduction to Current Research, Wiley, New York (1962).
3. J.D. Brown and K.V. Kuchař, *Dust as a standard of space and time in canonical quantum gravity*, Phys. Rev. D **51**, 5600 (1995).
4. A. Connes and C. Rovelli, *Von Neumann algebra automorphisms and time-thermodynamics relation in general covariant quantum theories*, Class. Quantum Grav. **11**, 2899 (1994).

The essence of the problem:

The unitary evolution of any QFT is governed by the Schrödinger type equation

$$i\hbar\partial_t\Psi = \hat{H}_{QFT}\Psi, \tag{1}$$

but, in the formally quantized GR, we have the Wheeler-DeWitt (WD) equation

$$-\hbar^2\kappa^2 G_{abcd}(x)\frac{\delta^2\Psi}{\delta g_{ab}(x)g_{cd}(x)} - \kappa^{-2}|g|^{1/2}(x)R(x)\Psi = 0, \tag{2}$$

$G_{abcd} := \frac{1}{2}|g|^{-1/2}(x)(g_{ac}(x)g_{bd}(x) + g_{bc}(x)g_{ad}(x) - g_{ab}(x)g_{cd}(x))$ is the DeWitt metric.
g_{ab} is the spatial part of the spacetime metric $g_{\mu\nu}$, $|g| := \det g_{ab}$.
R is the scalar curvature corresponding to g_{ab}.

Issues

1. It is not clear how to transform uniquely the WD equation to the form (1). Different time choices leads to different (unitary inequivalent) equations (1). Classically, such a transform is a canonical one, but in QFT it can be (and, in fact, is) a non-unitary transform.
2. If one represents the QG evolution in the form (1), then it is not clear whether this evolution respect the general covariance symmetry. Eq. (1) is in a $(3+1)$ form.
3. The (generalized) algebra of gauge constraints may possess anomalies due to the ordering problems.

On the classical side (one forgets about the ordering issues)

1. One can convert the Hamiltonian constraint (the classical analog of (2)) to the form of standard Hamilton equations with respect to some (arbitrary) time variable by a canonical transform. The classical evolutions corresponding to different time choices are related by the canonical transformations.
2. The general covariance is preserved by the Hamiltonian mechanics since it is just another representation of the classical Lagrangian equations of motion that are explicitly generally covariant.

Possible solutions

1. Introduce time before quantization. The classical constraints are solved introducing a certain time variable and then quantized:
   - a. Internal time. The time is a functional of canonical variables of the theory, e.g., the mean extrinsic curvature time.
   - b. Reference fluids. The time is identified with a parameter on the integral curves of the 4-velocity of a relativistic fluid. The nature and properties of this fluid are obscure. Several proposals were given: the Gaussian fluid (the fluid providing the Gauss gauge condition on the metric $g^{00}=1$, $g^{0a}=0$), the parameterized forms of the harmonic and unimodular gauges ($\det g_{ab}=-1$), the elastic medium, the dust.
2. Introduce time after quantization. The time variable is somehow defined using the solution $\Psi[g]$ of the WD equation so that to provide the positive definiteness of probability:
   - a. Klein-Gordon interpretation of the WD equation.
   - b. Third quantization.
   - c. Semiclassical approximation based on the Hamilton-Jacobi equation for the WD equation.
3. There is no distinguished time variable at all:
   - a. Conditional probability approach.
   - b. Consistent histories approach.

Shortcomings of all the solutions proposed

- All the schemes referred above does not take seriously into account the non-commutativity of operators in QFT. At best, some of these schemes are realized in a minisuperspace. The representation of the field operators in a certain Fock space is entirely disregarded.
- Eventually, it is not clear how to implement these schemes to calculate any observable (the averages or amplitudes) in QFT.

Additional shortcomings

1. Approach 1a). It is hard to find such an intrinsic variable that provides a timelike direction of time for the metric of a general form. It is not clear whether the general covariance holds in this case or not, and certain plausible arguments can be adduced that it does not.
2. Approach 1b). It is not clear what the “equation of state” should be used for this fluid. Besides, the general covariance is lost in terms of the initial set of fields.

Note

- Approach 1b) is the closest one to that we evolve. The difference consists in that we prove that this fluid not only can be introduced, but must be introduced. Furthermore, we obtain a certain class equations of state for this fluid and give a concrete scheme how to calculated the observables. The general covariance is restored by quantizing the reference fluid and including it into the set of dynamical fields of the theory.

Additional shortcomings (continuation)

3. Approach 2a). It encounters all the problems with probability interpretation of the usual Klein-Gordon equation on a non-stationary background.
4. Approach 2b). The physical interpretation and a rigorous mathematical formulation are unclear.
5. Approach 2c). It is not clear how to work with the solutions to the WD equation that are not quasiclassical $\Psi \neq Ae^{iS}$, or given by a superposition of the quasiclassical solutions.
6. Approach 3a). It represents a significant departure from the conventional quantum theory. The rigorous mathematical apparatus for applications in QG is not developed and apparently cannot be developed. Besides, it needs an internal time (see point 1a)).
7. Approach 3b). It also represents a modification of the rules of conventional quantum theory. The rigorous mathematical formalism is not developed and, at the present moment, the approach is nothing but a good intention.

Recommended literature on the EA and background field method

1. B.S. DeWitt, *The Global Approach to Quantum Field Theory* Vol. 1,2 (Clarendon, Oxford, 2003).
2. I.L. Buchbinder, S.D. Odintsov, and I.L. Shapiro, *Effective Action in Quantum Gravity* (IOP, Bristol, 1992).
3. S. Weinberg, *The Quantum Theory of Fields*. Vol. 2: *Modern Applications* (CUP, Cambridge, 1996).

Generating functional of n-point Green functions

$$\begin{aligned} Z(K) = e^{iW(K)} &= \langle out|\,\mathrm{Texp}\left\{i\int_{t_{in}}^{t_{out}} d\tau K(\tau)\phi(\tau)\right\}|in\rangle = \\ &= \langle out, t_{out}|\,\mathrm{Texp}\left\{-i\int_{t_{in}}^{t_{out}} d\tau[\mathcal{H}(\tau,\phi)-K(\tau)\phi]\right\}|in, t_{in}\rangle = \\ &= \sum_{n=0}^{\infty}\int d\tau_1\ldots d\tau_n \frac{\delta^n Z}{\delta K(\tau_1)\cdots\delta K(\tau_n)}\Big|_{K=0}\frac{K(\tau_1)\cdots K(\tau_n)}{n!} \equiv \sum_{n=0}^{\infty} Z_n\frac{K^n}{n!}, \end{aligned} \tag{3}$$

ϕ is the set of fields, K are the sources, Z_n are the n-point Green functions, W_n are the connected ones, $\mathcal{H}$ is the Hamiltonian of the theory,
$|in\rangle$ and $|out\rangle$ are the in- and out-vacua in the Heisenberg representation,
$|in, t_{in}\rangle$ and $|out, t_{out}\rangle$ are the same vacua in the Schrödinger representation.

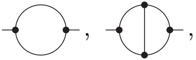
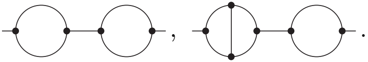
Effective action (EA)

Generating functional of the one-particle irreducible (1PI) n-point Green functions

$$\Gamma(\bar{\phi}) := \big(W(K) - K\bar{\phi}\big)_{K=-\Gamma_1(\bar{\phi})}, \qquad K = -\Gamma_1(\bar{\phi}), \qquad \Gamma_2 = -W_2^{-1}. \tag{4}$$

$\Gamma(\bar{\phi})$ is the Legendre transform of the generating functional of the connected n-point Green functions.

Some $1PI$ and non $1PI$ contributions to Γ_2 in ϕ^3 theory

1PI:  non 1PI:  (5)

Remark

- If one knows the EA functional, one knows all the Green functions and, consequently, the whole QFT.
- W_n is a tree consisting of $W_2 = -\Gamma_2^{-1}$ and $\Gamma_{k\leq n}$. Hence, the divergencies do not appear at the stage of reconstruction of W_n from $\Gamma_{k\leq n}$.
- All the divergencies and the counterterms to them can be analyzed on the level of the EA.

Background field method

$$e^{i\Gamma(\bar\phi)} = \langle out|\,\mathrm{Texp}\big[-i(\hat\phi-\bar\phi)\Gamma_1|in\rangle = \langle\widetilde{out}|\,\mathrm{Texp}(-i\hat\phi\Gamma_1)|\widetilde{in}\rangle = e^{i\Gamma_{\bar\phi}[0]}. \qquad (6)$$

$\Gamma_1 \equiv \delta\Gamma[\bar\phi]/\delta\bar\phi = \Gamma_{\bar\phi}[\tilde\phi]/\delta\tilde\phi|_{\tilde\phi=0}$, $\Gamma_{\bar\phi}[0]$ is the vacuum effective action on the background $\bar\phi$.
In the second equality the unitary transform $\hat\phi \to \hat\phi + \bar\phi$ has been performed.

Note

- In order to find the EA, it is sufficient to calculate the vacuum EA on an arbitrary fixed background $\bar\phi$.

Some vacuum $1PI$ in ϕ^3 theory

(7)

Here the bold lines and vertices are the propagators and vertices on the background $\bar\phi$.

- If one expands the propagators and vertices in (7) in a functional Taylor series in $\bar\phi$, one obtains, in particular, the 1PI diagrams (5).

Background field method

Gauge transformations

$$\delta_\varepsilon \phi^\mu = \varepsilon^\alpha R^\mu_\alpha(\phi), \qquad \delta_\varepsilon S(\phi) = \varepsilon^\alpha R^\mu_\alpha \frac{\delta S}{\delta \phi^\mu} \equiv 0,$$
$$\text{YM: } \delta_\varepsilon A^a_\mu = \partial_\mu \varepsilon^a + igf^a{}_{bc}\varepsilon^b A^c_\mu, \qquad \text{GR: } \delta_\varepsilon g_\mu = \mathcal{L}_\varepsilon g_{\mu\nu} = \nabla_{(\mu}\varepsilon_{\nu)}. \tag{8}$$

$S(\phi)$ is the classical action, $R^\mu_\alpha(\phi)$ are the generators of gauge transformations.
$f^a{}_{bc}$ are the structure constants of the Lie algebra.

Background field gauge: $\phi^\mu =: \bar\phi^\mu + \tilde\phi^\mu$, $\bar\phi^\mu$ is the background field.

$$\delta_\varepsilon(\bar\phi^\mu + \tilde\phi^\mu) = \delta_\varepsilon \tilde\phi^\mu = \varepsilon^\alpha R^\mu_\alpha[\bar\phi + \tilde\phi], \quad \text{linear gauge: } \chi^\alpha = P^\alpha_\mu(\bar\phi)\tilde\phi^\mu. \tag{9}$$

- If $R^\mu_\alpha[\phi] = t^\mu_\alpha + r^\mu_{\alpha\nu}\phi^\nu$ and R^μ_α are the generators of the Lie algebra (this holds for YM and GR, but not for SUGRA) then one may also realize (9) as
$$\delta_\varepsilon \bar\phi^\mu = \varepsilon^\alpha R^\mu_\alpha(\bar\phi), \qquad \delta_\varepsilon \tilde\phi^\mu = \varepsilon^\alpha \tilde\phi^\nu \partial_\nu R^\mu_\alpha, \text{i.e., } \mathcal{L}_{\varepsilon^\alpha R_\alpha(\bar\phi)} \tilde\phi^\mu = 0. \tag{10}$$
The field $\tilde{\phi^\mu}$ is covariantly transformed under the background gauge transformations.
- If also the gauge fixing operator $P^\alpha_\mu(\bar\phi)$ is chosen to transform covariantly under (10), e.g.,
$$\text{YM: } \chi^\alpha = \bar\nabla^\mu \tilde A^a_\mu = \partial^\mu \tilde A^a_\mu + igf^a{}_{bc}\bar A^{b\mu}\tilde A^c_\mu, \qquad \text{GR: } \chi^\alpha = \bar\nabla^\sigma \tilde g_{\mu\sigma} - \tfrac{1}{2}\bar g^{\lambda\sigma}\bar\nabla_\mu \tilde g_{\lambda\sigma}, \tag{11}$$
then the construction (6) for the EA with the Faddeev-Popov recipe for the gauge fixing is explicitly invariant under the background gauge transformations
$$\delta_\varepsilon \Gamma(\bar\phi) = \varepsilon^\alpha R^\mu_\alpha(\bar\phi)\frac{\delta\Gamma(\bar\phi)}{\delta\bar\phi^\mu} \equiv 0 \text{ (an infinite set of the Ward identities).} \tag{12}$$

Recommended literature on the quantum anomalies

1. S. Weinberg, *The Quantum Theory of Fields*. Vol. 2: *Modern Applications* (CUP, Cambridge, 1996).
2. M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, New York, 1995).

Any symmetry transform

$$\delta_\varepsilon \Gamma(\bar{\phi}) = \varepsilon^\alpha R^\mu_\alpha(\bar{\phi}) \frac{\delta \Gamma(\bar{\phi})}{\delta \bar{\phi}^\mu} \equiv 0. \tag{13}$$

$R^\mu_\alpha(\phi)$ are the (global or local) symmetry generators.

Quantum anomaly

$$\delta_\varepsilon S(\phi) \equiv 0, \text{but } \delta_\varepsilon \Gamma(\bar{\phi}) \not\equiv 0. \tag{14}$$

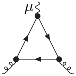
Remark

- The root of the anomalies is the non-commutativity of QFT operators which is also accompanied by the divergencies.
- The quantum anomalies of the global symmetries are admissible, i.e., QFT is consistent even when these anomalies take place.
- The quantum anomalies of the local symmetries are inadmissible and should be avoided in some way. Otherwise, QFT is inconsistent.

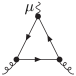
Symmetries of the EA and anomalies

Perturbative chiral anomalies: classically, $\partial_\mu j^{5\mu} = 0$, but

$$\partial_\mu \langle j^{5\mu}(x_1) j^\nu(x_2) j^\rho(x_3) \rangle \sim \partial_\mu \Bigg($$



$$+$$



$$\Bigg) \neq 0. \tag{15}$$

Chiral anomalies

1. In QFTs with chiral fermions, the conservation of one of the currents entering the triangle diagram is violated by the quantum corrections provided the Lorentz invariance holds (S. Adler, 1969; J.S. Bell, R. Jackiw, 1969).
2. This statement is also valid when one of the currents (apart from the chiral one) is replaced by the energy-momentum tensor (R. Delbourgo, A. Salam, 1972).
3. In QFTs with chiral fermions in the spacetime dimension $d = 4k + 2$, $k = 0, 1, \ldots$, there is a purely gravitational anomaly when all the currents in the one-loop diagram are the energy-momentum tensors (L. Alvarez-Gaumé, E. Witten, 1984). So, the general covariance is violated in these theories whenever the numerical coefficient at this anomaly is not zero. The cancelation may happen due to the contributions to this anomaly made by the fields with different spins.
4. In the standard model (SM), the chiral symmetry is the gauge one and so the consistency requires that all the chiral anomalies should be canceled. This occurs when

$$\mathrm{tr}(\gamma^5 \lambda^a \{\lambda^b, \lambda^c\}) = 0 \text{ (chiral currents)}, \quad \mathrm{tr}(\gamma^5 \lambda^a) = 0 \text{ (ch. curr. + the e.-m. tens.)}. \tag{16}$$

   $\gamma^5 = +1$ is for the right fermions and $\gamma^5 = -1$ is for the left fermions.
   λ^a are the generators of the gauge symmetry group in the corresponding representation.

5. The spectrum of particles in the SM is such that all the chiral gauge anomalies cancel.

Recommended literature on QFT on a curved background

1. A.A. Grib, S.G. Mamaev, and V.M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Lab. Publ., St. Petersburg, 1994) [Russian eds. 1980, 1988].
2. B.S. DeWitt, *The Global Approach to Quantum Field Theory* Vol. 1,2 (Clarendon, Oxford, 2003).
3. B.S. DeWitt, *Quantum field theory in curved spacetime*, Phys. Rep. **19**, 295 (1975).
4. V.P. Frolov and I.D. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Springer, New York, 1998).
5. N.D. Birrel and P.C.W. Davies, *Quantum Fields in Curved Space* (CUP, Cambridge, 1982).

Covariant formulation of QG

- QFT on a curved background is constructed using the standard background field method that we have already considered. Therefore, it allows one to find the EA by the explicit calculations not only for the matter fields but also for the gravitons. In fact, this is QG in the so-called covariant formulation.
- This approach, when properly applied, tacitly implies all the basic principles of QFT and GR and, in that sense, is canonical.

Problems of the covariant approach

1. QG is non-renormalizable and so possesses a less predictive power than, say, the SM.
2. The EA (and observables) depends on the choice of the Hamiltonian of quantum fields, the corresponding vacuum state, and the creation-annihilation operators for particles. This is the reincarnation of the problem of time in canonical QG. Contrary to the Minkowski spacetime, there is no privileged vacuum state for a general background metric that does not have the Killing vectors.

$$S[\phi] = \int d^4x \sqrt{|g|}\mathcal{L} \;\Rightarrow\; \pi := \sqrt{|g|}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \quad \dot{\phi} := \xi^\mu \partial_\mu \phi \;\Rightarrow\; H[\xi] = \int d\mathbf{x}(\pi\dot{\phi} - \sqrt{|g|}\mathcal{L}). \tag{17}$$

- The classical evolutions generated by the different Hamiltonians $H[\xi]$ are related by the canonical transformations, but QFTs corresponding to the different Hamiltonians $H[\xi]$ can be unitary inequivalent.

Problems of the covariant approach (continuation)

3. For a non-stationary background metric, the notion of a particle is ambiguous. The definitions of particles in the reference frames related by a nonlinear transform affecting the time variable differ and, in many cases, are not related by a unitary transform (S.A. Fulling, 1973; W.G. Unruh, 1976). This problem is related to the previous one.

Problems of the covariant approach (continuation)

4. As we saw, formally, the background field method gives

$$\sqrt{|g|}T^{\mu\nu} := -2\delta\Gamma/\delta g_{\mu\nu}, \qquad \nabla_\mu T^{\mu\nu} \equiv 0 \text{ (the Ward identities)}, \tag{18}$$

but since the EA depends on the external structure (the time-like vector field ξ^μ) defining the Hamiltonian and the vacuum state, the general covariance (the background independence) and, hence, the Ward identities are violated by quantum corrections (S. Hawking, 1970; A.A. Grib, B.A. Levitskii, V.M. Mostepanenko, 1974). That is, the quantum gravitational anomaly arises. For special background metrics (stationary metrics, FLRW, (A)dS), there is a distinguished "time direction" that allows one to define the natural Hamiltonian, the vacuum state, etc. In that case, the first Ward identity,

$$\nabla_\mu T^{\mu\nu} = 0 \text{ (on the solutions to the equations of motion)}, \tag{19}$$

is preserved (V.P. Frolov, A.I. Zel'nikov, 1987), but the higher Ward identities, which are obtained by variation of (18) with respect to the metric, are still broken. The prove of the fact that the anomaly of this type does exist and cannot be canceled by the counterterms to the EA encounters severe technical issues that were solved only recently (I.S. Kalinichenko, P.O. Kazinski, 2014).

5. The explicit calculations of the vacuum energy-momentum tensor on stationary metric backgrounds show that it contains not only the covariant combinations of the metric and its derivatives (the curvatures), but also the structures involving the Killing vector defining the Hamiltonian (D.N. Page, 1982; K.W. Howard, 1984; M.R. Brown, A.C. Ottewill, D.N. Page, 1986; in the explicit form it was pointed out in V.P. Frolov, A.I. Zel'nikov, 1987).

QFT on a curved background

Résumé

- Virtually, the covariant approach to QG possesses two main problems 1 and 2. Issues 3,4, and 5 are the consequences of problem 2.

Possible solutions to problem 2

1. The dependence on the choice of vacuum state is essentially non-perturbative (keep in mind the example of a perturbation theory for a quantum particle near the false vacuum) and, at any finite order of the perturbation theory, the dependence of EA on the vector field ξ^μ can be eliminated by the counterterms. So, one may try to solve the problem just by defining QG as a perturbative series over the Minkowski metric (the standard derivative expansion in the Riemann normal coordinates is of that type solution). However, the perturbative series is asymptotic and the neglect of the non-perturbative terms leads eventually to non-unitary dynamics. Moreover, in that case, the gravitational field entering the one-particle quantum equations (Klein-Gordon, Dirac, Maxwell) must be treated only perturbatively, too. This approach represents a significant departure from the conventional quantum theory.
2. Continue analytically the theory to the Riemann spacetime (with the positive definite metric). Then, imposing zero boundary conditions at spacetime infinity, one obtains a unique propagator and Euclidean QFT. However, the result of this analytical continuation depends on a choice of the time variable which becomes complex. For non-static spacetimes, the relation between the initial QFT and the Euclidean QFT so constructed is unclear.

Possible solutions to problem 2 (continuation)

3. Introduce the preferred vector field ξ^μ by hand from some physical (symmetrical) reasonings. This is the standard solution used in the literature for the special background metrics (Schwarzschild, Kerr, FLRW...). For a general metric, such an approach was proposed and developed in (A.A. Grib, S.G. Mamaev, 1969). However, as we have discussed, this approach violates the general covariance Ward identities, and the resulting QFT is inconsistent. The quantum gravitational anomaly arises.

Quantum gravitational anomaly

- The EA contains the finite terms of the form (I.S. Kalinichenko, P.O. Kazinski, 2014)

$$\Gamma \sim \operatorname{Re} e^{-am/m_g}. \tag{20}$$

  m is the mass of a particle, say, the electron mass. a is a certain complex constant. m_g is a scalar combination of the vector field ξ^μ, the metric, and their derivatives. m_g cannot be expressed in terms of the metric alone.
- For the usual gravitational fields, these terms are very tiny $\sim 10^{-20}\Lambda_c$, Λ_c is the cosmological constant, but the fact of their existence is of paramount importance.
- These contributions cannot be canceled by the counterterms (e.g., N.N. Bogolyubov, D.V. Shirkov, 1980; J.C. Collins, 1984) as long as they are non-polynomial in momenta.
- The existence of such terms was expected long ago (e.g., A.I. Zel'nikov, 1984; S.P. Gavrilov, D.M. Gitman, 1996), but was not proved until recently.

Possible solution to the gravitational anomaly problem

1. The quantum gravitational anomaly cannot be canceled by the contributions of different types of particles in the SM due to its non-trivial dependence on the mass of a particle. However, if we make the vector field dynamical, include it into the set of fields of the theory, and quantize, then the gravitational anomaly is canceled out and the general covariance is restored. As a result, the additional degrees of freedom – the quantum vector field ξ^μ – appear in the model.
2. Note that this vector field is not simply another one quantum field of the model, but its average must by used to define the Hamiltonian according to the approach 3 above.

Dynamics of the vector field ξ^μ

1. The quantum dynamics of the vector field ξ^μ is dictated by the Ward identities fulfillment (P.O. Kazinski, 2011)

$$\nabla^\nu T_{\mu\nu} = \mathcal{L}_\xi \Gamma_\mu + \nabla^\nu \xi_\nu \Gamma_\mu = 0, \qquad \sqrt{|g|} T^{\mu\nu} := -2\delta\Gamma/\delta g_{\mu\nu}, \quad \sqrt{|g|}\Gamma_\mu := -\delta\Gamma/\delta\xi^\mu.$$

2. The evolution of ξ^μ obeys the Euler equations for a relativistic perfect isentropic fluid

$$\mathcal{L}_\xi \Gamma_\mu + \nabla^\nu \xi_\nu \Gamma_\mu = 0 \;\Leftrightarrow\; \nabla_\mu(\xi^\mu w) = 0, \quad \xi^\mu \nabla_{[\mu}(\Gamma_{\nu]}/w) = \mathcal{L}_\xi(\Gamma_\nu/w) = 0. \tag{21}$$

$w := \xi^\mu \Gamma_\mu$ is the enthalpy density. $\beta := \sqrt{\xi^2}$ is the reciprocal temperature.
$\sigma := \beta w$ is the entropy density. The total entropy is conserved.
The equation of state of the fluid is determined by the EA.

Fock-Taub formalism for relativistic hydrodynamics

The Fock-Taub action for a perfect isentropic fluid

$$S[x(\varkappa)] = \int_N d\varkappa \sqrt{|h|} p[\rho^a, h_{ab}]. \tag{22}$$

$\varkappa = \{\tau, \sigma^i\}$, $i = \overline{1,3}$.
$x^\mu(\varkappa)$, $\mu = \overline{0,3}$, is a map of the 4-dimensional manifold N to a 4-dimensional spacetime M.
$\rho^a(\varkappa)$, $a = \overline{0,3}$, is a vector field on N such that $\rho^a h_{ab} \rho^b > 0$. ρ^a is not a dynamical field.
$h_{ab} := \partial_a x^\mu \partial_b x^\nu g_{\mu\nu}$ is the induced metric on N.
$g_{\mu\nu}$ is the spacetime metric.
p is some scalar constructed in terms of ρ^a, h_{ab}, and their derivatives. It is the fluid pressure.

Equations of motion

$$\frac{\delta S}{\delta x^\mu(\varkappa)} = \sqrt{|h|} \nabla_\lambda T^\lambda_\mu = 0, \qquad \sqrt{|g|} T^{\mu\nu} = -2 \frac{\delta S}{\delta g_{\mu\nu}}. \tag{23}$$

General features

- The Fock-Taub formalism is based on the so-called relativistic Lagrangian representation of a fluid.
- In our case, $\xi^\mu = \rho^a \partial_a x^\mu$, and the pressure p is the classical part of the EA involving the vector field ξ^μ.
- Equations of motion (23) reproduces the ones (21) following from the Ward identity.

Normalization conditions on the EA

- Since the diagrams defining the EA contain divergencies, some natural normalization conditions should be imposed on it to fix the renormalization ambiguity.
- The normalization conditions are dictated by the symmetry requirements and the experimental data. The typical examples of the normalization conditions are the physical mass and charge normalizations in QED.
- It is important at this stage that the general covariance holds only if the vector field ξ^μ is included into the EA.

Natural normalization conditions

1. The Lorentz-invariance. For the flat spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$, the dependence of the EA on the field ξ^μ should disappear;
2. The compliance with the Einstein equations. In the flat spacetime limit, the vacuum expectation value of the energy-momentum tensor operator of the matter fields and the field ξ^μ must be zero;
3. Minimality. The initial classical action of the field ξ^μ contains only those structures that arise as divergencies in calculating the quantum corrections using the physical regularization (e.g., J. Collins, A. Perez, D. Sudarsky, L. Urrutia, H. Vucetich, 2004) by the energy cutoff.

Fluid pressure

General form of the pressure without higher derivatives as follows from condition 3

$$p(\rho^2) = \sum_{k=0}^{2} \sum_{l=0}^{\infty} a_{kl} t^{2k} \ln^l t^2. \tag{24}$$

a_{kl} are the curvature independent gauge invariant scalars of mass dimension 4.
$t_a := \rho_a/\rho^2$.

- The vector field ρ^a is normalized in such a way that $\rho^2 = 1$ in the flat spacetime.
- The logarithmic corrections describe the so-called anomalous scaling and are assumed to be small. In the weak field limit, $\rho^2 \approx 1$ that additionally diminishes these corrections.
- a_{kl} depend on the Higgs field, but this will not be relevant for our discussion.

Energy-momentum tensor

$$T^{ab} = -2p'\rho^a\rho^b - ph^{ab} = wu^a u^b - ph^{ab}. \tag{25}$$

The prime denotes the derivative with respect to ρ^2.
$u^a := \rho^a/\sqrt{\rho^2}$.

Normalization conditions as follows from conditions 1 and 2

$$p(\rho^2)\big|_{\rho^2=1} = 0, \qquad p'(\rho^2)\big|_{\rho^2=1} = 0. \tag{26}$$

The simplest model

$$p = -\frac{b}{2}(t^2 - 1)^2. \tag{27}$$

b is a negative constant of mass dimension 4.

General features

- The simplest mean to quantize relativistic hydrodynamics is to use the background field method which states that the knowledge of quantum evolution of small fluctuations over an arbitrary background is sufficient to reconstruct the whole quantum dynamics.
- The Fock-Taub action is highly nonlinear and so the corresponding QFT is perturbatively non-renormalizable. It should be treated in the effective field theories framework (S. Endlich, A. Nicolis, R. Rattazzi, J. Wang, 2011).
- The perturbative non-renormalizability of quantum relativistic hydrodynamics reduces its predictive power. Nevertheless, it can be used to derive certain predictions in QG.
- In this way, the self-consistent model of QG is constructed though it is non-renormalizable.
- One of the predictions of this model is the existence of a relativistic fluid with the properties resembling the properties of DM.

General references on the DM and its explanation

1. P. Schneider, *Extragalactic Astronomy and Cosmology* (Springer, Heidelberg, 2015).
2. D.S. Gorbunov and V.A. Rubakov, *Introduction to the Theory of the Early Universe* Vol. 1,2 (World Scientific, London, 2011).
3. S. Nojiri and S.D. Odintsov, *Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models*, Phys. Rep. **505**, 59 (2011).
4. B. Famaey and S.S. McGaugh, *Modified Newtonian dynamics (MOND): Observational phenomenology and relativistic extensions*, Living Rev. Relativity **15**, 10 (2012).

Evidences for the DM

1. CMB anisotropy spectrum. To reproduce correctly the CMB anisotropy spectrum, one needs the CDM consisting of the unknown particles.
2. Structure formation of the Universe. The density fluctuations grow too slowly without the DM and so we would not have the observed large scale structure at the present moment.

Evidences for the DM (continuation)

3. Gravitational lensing. According to GR, the light rays are bent by a massive object at the angles depending on the mass of the object. The mass of galaxy clusters measured by means of this effect is much larger than the mass of the luminous matter in these clusters.
4. X-ray observations of a hot gas in clusters. X-ray observations of a hot gas in clusters provide an alternative way to estimate the mass of a cluster. The mass of clusters estimated by this means is larger than that of the ordinary barionic matter by a factor of 5.
5. Virial theorem. The virial theorem applied to the galaxy clusters implies that their mass is mainly due to the DM. The values of masses of galaxy clusters obtained by methods 3, 4, 5 agree by the order of magnitude.
6. Observations of the colliding clusters. The observations of the colliding galaxy clusters using the gravitational lensing technique clearly show that the DM distribution does not always follow the luminous matter distribution and behaves as (almost) collisionless gas of particles. This, in particular, implies that the DM possesses its own degrees of freedom.
7. Star dynamics in galaxies. The galaxy rotation curves and the star dynamics in galaxies evidence for the existence of the DM.

Some possible explanation of the DM phenomenon

1. The common point of view is that the DM is a pressureless gas of weakly interacting massive particles (WIMPs).
2. The DM is a manifestation of deviations of the gravitational laws from that dictated by General Relativity ($f(R)$ models and the theories with higher derivatives).
3. Modified Newton dynamics (MOND). This approach postulates a modification of dynamics at small accelerations or introduces the additional tensor fields into the model.

Shortcomings

1. Approach 1. Due to the pressureless property of the CDM, the DM density profile in galaxies possesses a cusp at the galaxy center which is not observed in the astrophysical data. WIMPs are not detected yet and the natural candidates for them are absent at the present moment. The predictions of the (natural) MSSM providing several candidates for WIMPs were not confirmed at LHC. Besides, SUSY introduces many other particles and, in that sense, this solution to the DM problem is not minimal.
2. Approach 2. The viable variants of these theories are equivalent to the scalar-tensor theories of gravity. They can reproduce the correct cosmological evolution, but cannot be responsible for the DM on the scales of galaxy clusters.
3. Approach 3. In its orthodox form, MOND forbids the DM to have its own dynamics. Besides, this model violates the Lorentz-invariance and the general covariance, or involves a huge number of additional degrees of freedom (TeVeS). Even TeVeS cannot explain the dynamics of DM in the colliding clusters. The consistency of its quantum version is unclear.

CDM from the quantum gravitational anomaly

Normalization conditions 1, 2 (26) imply in the limit of weak slowly varying gravitational fields

$$w \approx \varepsilon \approx \sigma \sim x, \quad p \sim x^2, \qquad x := 1 - \rho^2 = 1 - \xi^2 \ll 1. \tag{28}$$

$\varepsilon = w - p$ is the energy density of a fluid.

One can adjust the constants in p in such a way that the first n derivatives of p with respect to ρ^2 at $\rho^2 = 1$ vanish

$$\begin{gathered}\varepsilon \approx w = -2\beta^2 p' \approx 2Ax^n, \qquad p \approx A\frac{x^{n+1}}{n+1}, \quad n \in \mathbb{N}, \\ p = \frac{\varepsilon^{1+1/n}}{2(n+1)(2A)^{1/n}} =: K\varepsilon^{1+1/n} = \frac{\varepsilon x}{2(n+1)}.\end{gathered} \tag{29}$$

A is a positive constant with the mass dimension 4.

Polytrope

- (29) is the equation of a polytrope with the index n and the polytropic constant K. The constants n and K are universal.
- If one identifies $\Lambda_{dm} x/[2(n+1)]$ with kT, the equation of state (29) coincides with the equation of state for a perfect gas.
- The limit $n \to \infty$ corresponds to the isothermal polytropic process.

Parameters of the polytropic equation of state

The constant A can be estimated from the local DM observations (J.I. Read, 2014). In this case, one sets $x \approx -2\varphi_\odot$ and takes

$$|\varphi_\odot| \sim 10^{-6}, \qquad \varepsilon_\odot \sim 0.5\,\text{GeV/cm}^3, \\ A = \frac{\varepsilon_\odot}{2|2\varphi_\odot|^n}. \tag{30}$$

Some values of A for several different n

$$\begin{aligned} A_1 &\approx 1.25 \times 10^5\,\text{GeV/cm}^3 \approx (1\,\text{eV})^4, \\ A_4 &\approx 1.6 \times 10^{22}\,\text{GeV/cm}^3 \approx (19\,\text{keV})^4, \qquad \text{cm}^{-3} = 7.7 \times 10^{-42}\,\text{GeV}^3 \\ A_5 &\approx 7.8 \times 10^{27}\,\text{GeV/cm}^3 \approx (0.5\,\text{MeV})^4, \\ A_9 &\approx 4.9 \times 10^{50}\,\text{GeV/cm}^3 \approx (248\,\text{GeV})^4. \end{aligned} \tag{31}$$

Polytropic index

- To avoid the cusp problem of the standard DM dust model, the polytropic fluids were already considered, suggesting that

$3.5 \leq n \leq 5$ (C.J. Saxton, I. Ferreras, 2010; C.J. Saxton, 2013; ...),

$n \approx 5$ (L.G. Cabral-Rosetti, et al., 2004; J. Calvo, et al., 2009),

$n = 3$ (A. Balaguera-Antolínez, D.F. Mota, M. Nowakowski, 2007),

$n = 1$ (P.J.E. Peebles, 2000; J. Goodman, 2000; T. Harko, G. Mocanu, 2012; ...).

Cosmological solution

- Let us check that the polytropic fluid does provide a model of the CDM.

Assume that the reference fluid is at rest on average and its density is homogeneous

$$\xi^\mu = (\xi^0(t), 0, 0, 0) \quad \text{and} \quad ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j,$$
$$\varepsilon_{dm} = \Omega_{dm}\varepsilon_c \sim 10^{-6}\ \text{GeV/cm}^3, \qquad \varepsilon_c = 0.53 \times 10^{-5}\ \text{GeV/cm}^3, \tag{32}$$

ds^2 is the interval squared of the FLRW metric. ε_c is the critical energy density.
$\Omega_{dm} = 0.20$ is the fraction of the DM in the energy content of the present Universe.

- Therefore, we can consider that the fluid is in a weak field regime $\xi^2 \approx 1$ and obeys the polytropic equation of state (29).

Solution to the equation of covariant divergenceless of the energy-momentum tensor

$$\varepsilon = \varepsilon_{dm}\Big(\frac{a_0}{a}\Big)^3 \Big\{1 + K\varepsilon_{dm}^{1/n}\Big[1 - \Big(\frac{a_0}{a}\Big)^{3/n}\Big]\Big\}^{-n} \approx 2Ax^n, \tag{33}$$

a_0 is the present-day scale factor. Solution (33) corresponds to the CDM so long as $K\varepsilon_{dm}^{1/n} \ll 1$.
From (29) and (30), one infers

$$K = \frac{(2A)^{-1/n}}{2(n+1)} = \frac{|\varphi_\odot|\varepsilon_\odot^{-1/n}}{n+1}, \qquad K\varepsilon_{dm}^{1/n} = \frac{|\varphi_\odot|}{n+1}\Big(\frac{\varepsilon_{dm}}{\varepsilon_\odot}\Big)^{1/n}. \tag{34}$$

- From (30), (32), it is evident that $K\varepsilon_{dm}^{1/n} \ll 10^{-6}$ for any natural number n.

The cosmological value of $\xi^2(t)$ as follows from (33)

$$x \approx \Big(\frac{\varepsilon_{dm}}{2A}\Big)^{1/n}\Big(\frac{a_0}{a}\Big)^{3/n} = 2(n+1)K\varepsilon_{dm}^{1/n}\Big(\frac{a_0}{a}\Big)^{3/n} = 2|\varphi_\odot|\Big(\frac{\varepsilon_{dm}}{\varepsilon_\odot}\Big)^{1/n}\Big(\frac{a_0}{a}\Big)^{3/n}. \tag{35}$$

Summary

1. The relativistic fluid stemming from the quantum gravitational anomaly can be responsible for a considerable part of the CDM.
2. In the weak filed limit, its equation of state is described by a polytrope characterized by the two universal constants.
3. A virtual lack of reliable experimental data leaves a freedom in selection of the particular values of these constants.
4. The inclusion of this fluid into the model and its subsequent quantization provide a self-consistent model of QG (though non-renormalizable).
5. The fluid described by the field ξ^μ arises with necessity in QG provided the basic principles of QFT and GR remains intact on the quantum level, i.e., the general covariance is a fundamental symmetry of Nature.
6. The hypothesis about the identification of this fluid with a considerable part of the DM can be falsified (see point 2).