Neutron star superspace: quantifying the difference between stellar structures in general relativity and beyond

Arthur G. Suvorov

University of Tübingen

November 8, 2019,
Thessaloniki

General introduction

Neutron stars

Neutron stars represent some of the most extreme objects in the universe; very high densities $\rho_c \gtrsim 10^{15}$ g cm^{-3}, magnetic fields $B \gtrsim 10^{15}$G, and rotation rates $\nu \gtrsim 700$ Hz.

Gravity or matter?

With objects like black holes, its much more obvious that we tests of their nature (e.g. gravitational waves, X-ray reflection spectroscopy) are actually probing the theory of general relativity:

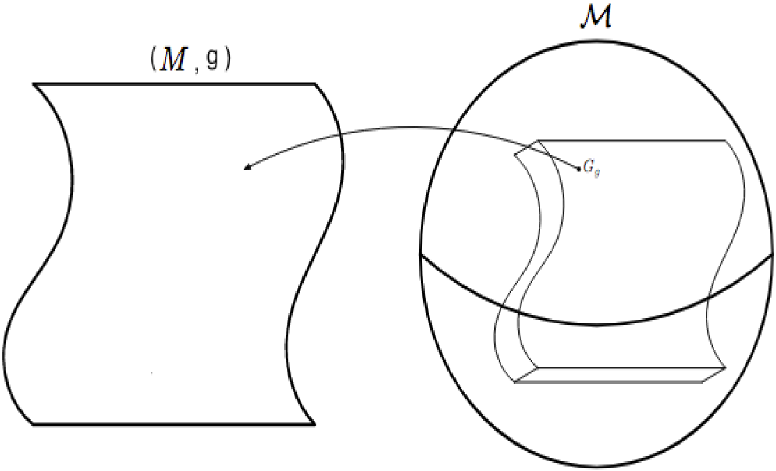
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{1}$$

In general relativity, the Kerr metric represents the unique geometry surrounding an isolated, rotating black hole in vacuum.

For neutron stars, not obvious!

PART 1: Superspace theory

EBERHARD KARLS UNIVERSITÄT TÜBINGEN



Geometrodynamics

"The stage on which the space of the Universe moves is certainly not space itself. Nobody can be a stage for himself; he has to have a larger arena in which to move.The arena in which space does its changing is not even the space-time of Einstein, for space-time is the history of space changing with time. The arena must be a larger object: Superspace... It is not endowed with three or four dimensions – its endowed with an infinite number of dimensions." (J.A. Wheeler: Superspace, Harpers Magazine, July 1974, p.9)

Connecting GR to classical mechanics

Configuration space

One think of the metric variables, that is the various components of g, as spanning a configuration space which defines the 'arena' of dynamics for GR.

As an example, consider a static, spherically symmetric spacetime (M, g) with:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2}$$

Rather than working on M and with the coordinates $\{t, r, \theta, \phi\}$, work on the space $\mathcal{Q} = \text{span}\{A, B\}$ and its tangent space $T\mathcal{Q} = \text{span}\{A, A', B, B'\}$.

GR as classical mechanics II

We thus have the identification

$$R_{\mu\nu} = 0 \iff 0 = \frac{\partial \mathcal{L}}{\partial q^j} - \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial \dot{q}^j}, \tag{3}$$

where

$$\mathcal{L} = R = R(q, q'), \tag{4}$$

where $q = A, B$ are the generalised coordinates defining the configuration space. In general R depends on q'' also, but we can use the divergence theorem to get rid of them if $\partial M = 0$.
$\implies$ The Einstein equations translate into Euler-Lagrange equations more commonly seen in classical mechanics.

The usefulness of this approach

Modified gravity

generating exact solutions via Noether symmetries with $L_X \mathcal{L} = 0$ (Cappoziello et al. 2006); or generalizing to axially symmetric spacetimes (Suvorov & Melatos 2017) to further the Ernst formalism of general relativity.

Quantum gravity

In Canonical Quantum Gravity, Superspace plays the role of the domain for the 'universe's wavefunction' which is subjected to the Wheeler-DeWitt equation; derivable from the Feynman path integral in the Euclidean quantum gravity paradigm: $Z = \int_C \mathrm{e}^{-A[g_{\mu\nu},\phi]} \mathcal{D}\mathbf{g}\, \mathcal{D}\phi$ with $\frac{\delta Z}{\delta N} = 0$.

Superspace: an (∞-dimensional) manifold

In general, the set of all Riemannian metrics over a Riemannian manifold M admits the structure of an infinite-dimensional (Fréchet) manifold): $\mathrm{Met}(M)$.

Points of $\mathrm{Met}(M)$ are Riemmanian metrics on M: each $p \in \mathrm{Met}(M)$ corresponds to a positive-definite, symmetric $(0,2)$-tensor over M.

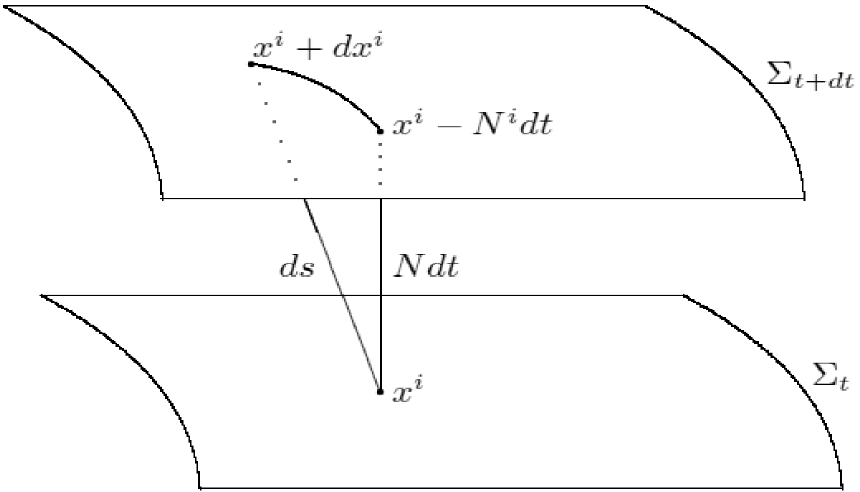
If M is compact, then one may introduce a metric, in the L^2-topology, over $\mathrm{Met}(M)$ as [Gil-Medrano & Michor, Q. J. Math **42**, 183 (1991)]

$$G(\alpha, \beta) = \int_M d^3x \sqrt{g} \left(g^{-1} \alpha g^{-1} \beta \right) , \tag{5}$$

where α and β are tangent vectors to the space of metrics at the 'point' g, which serves as a reference metric.

A measure of distance between two vectors naturally depends on the choice of basis $\{\mathbf{e}\}$ and the origin.

Arnowitt-Deser-Misner (ADM) $3+1$ spacetime split



We consider a finite dimensional submanifold with ($M = \Sigma$)!

$$G_{ij} = \int_M d^3x \sqrt{g}\, g^{nk} \frac{\partial g_{mn}}{\partial q^i} g^{\ell m} \frac{\partial g_{\ell k}}{\partial q^j}, \tag{6}$$

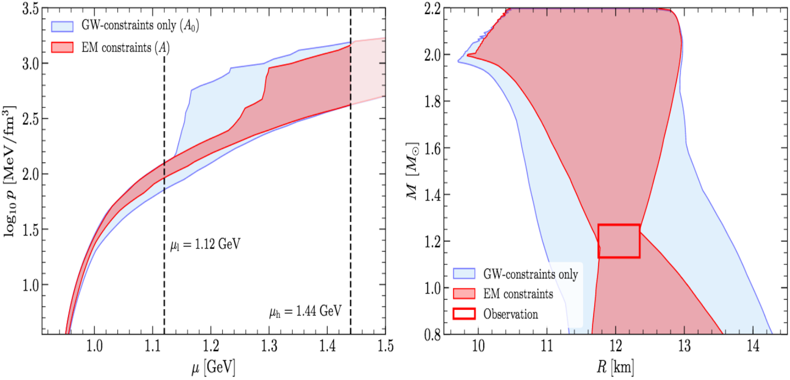
where $1 \leq i, j \leq N$. From (6), the relevant geometric quantities of $\mathrm{Met}_{\mathrm{NS}}(M)$ can be defined, including the Christoffel symbols Γ. The distance between two metrics h and k, described by parameters $\mathbf{q}_h$ and $\mathbf{q}_k$, respectively, is then given by the length of a geodesic $\gamma : [a, b] \mapsto \mathrm{Met}_{\mathrm{NS}}(M)$ connecting these points, viz.

$$d(h, k) = \int_a^b d\tau \sqrt{G_{ij} \frac{d\gamma^i}{d\tau} \frac{d\gamma^j}{d\tau}}, \tag{7}$$

for affine parameter τ, where $\gamma^j(a) = q_h^j$ and $\gamma^j(b) = q_k^j$, and $\boldsymbol{\gamma}$ satisfies the geodesic equation,

$$0 = \frac{d^2\gamma^i}{d\tau^2} + \Gamma^i_{jk} \frac{d\gamma^j}{d\tau} \frac{d\gamma^k}{d\tau}. \tag{8}$$

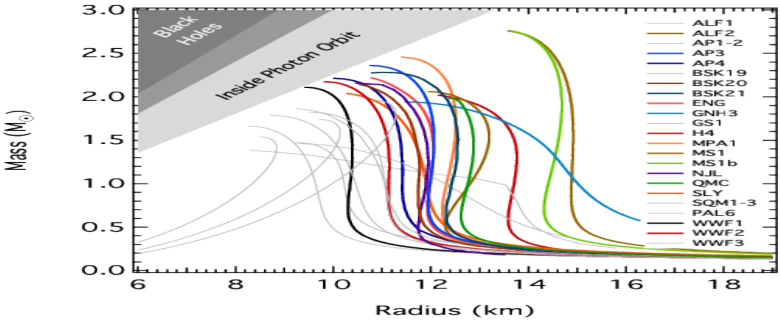
PART 2: Neutron star spacetimes



(Weih et al. 2019)

What do we know?

Neutron stars are extreme objects, and there provide us with a natural laboratory to probe physics we have no other way of investigating.
Our main uncertainty comes from the equation of state

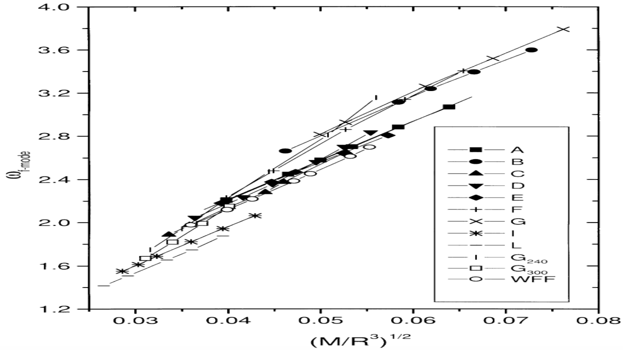


(Özel et al. 2016)

Some spacetimes/models are closer together than others.

Partial aside: Universal relations

Although mass-radius relations may be considerably different, other properties of neutron stars, such as the fundamental mode frequencies, might be almost the same for different equations of state.



(Andersson & Kokkotas 1998)

Superspace and neutron stars

What do we hope to learn from this? Although difficult to define $\mathrm{Met}_{\mathrm{NS}}(M)$ in total generality since, depending on the included physics, there may be an arbitrarily large (but finite) number of parameters which describe the stellar model; the stress-energy tensor may be arbitrarily complicated.

Nevertheless, suppose that a star can be described by N macroscopic parameters: $q^1, \ldots, q^N$, e.g. mass, radius, central temperature, polar magnetic field strength, rotational frequency, and so on. These parameters $\mathbf{q}$ define a natural coordinate basis for the N-dimensional space $\mathrm{Met}_{\mathrm{NS}}(M)$

$$\begin{aligned} G_{ij} =& 4\pi \int_0^{\bar{R}} dr \frac{r^2}{\sqrt{AB^3}} \Big[A \frac{\partial B}{\partial q^i} \left(B \frac{\partial A}{\partial q^j} + A \frac{\partial B}{\partial q^j} \right) \\ & + B \frac{\partial A}{\partial q^i} \left(3B \frac{\partial A}{\partial q^j} + A \frac{\partial B}{\partial q^j} \right) \Big]. \end{aligned} \tag{9}$$

Worked example

In natural units, the metric functions A and B for the Tolman VII ($\rho \propto R^2 - r^2$) metric read

$$A(r) = \left(1 - \frac{5M}{3R}\right) \cos^2\left[\Phi(r)\right], \tag{10}$$

and

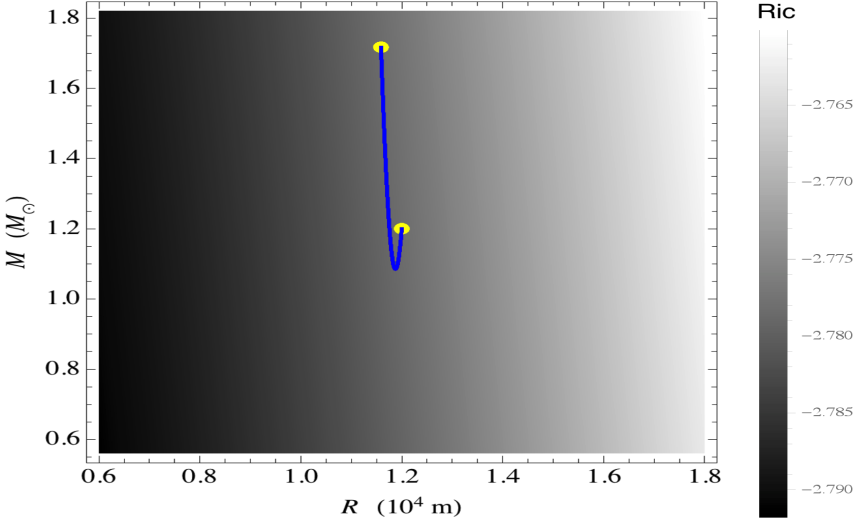
$$B(r) = \left[1 - \frac{Mr^2}{R^3}\left(5 - \frac{3r^2}{R^2}\right)\right]^{-1}, \tag{11}$$

where

$$\begin{aligned} \Phi(r) =& \frac{1}{2}\log\left[\frac{1 + 2\sqrt{\frac{3R}{M} - 6}}{\frac{6r^2}{R^2} - 5 + 2\sqrt{\frac{9r^4}{R^4} - \frac{15r^2}{R^2} + \frac{3R}{M}}}\right] \\ &+ \arctan\left[\frac{M}{\sqrt{3M\,(R - 2M)}}\right]. \end{aligned} \tag{12}$$

Outside of the star, $r > R$, match to the Schwarzschild exterior.

Tolman VII Geodesics



Some distances

Distances $d(R_1, M_1, R_2, M_2)$, defined in (7), between various Tolman VII configurations (10)–(12).

R_1 (10^4 m)	M_1 ($M_\odot$)	R_2 (10^4 m)	M_2 ($M_\odot$)	$d(R_1, M_1, R_2, M_2)$
1.35	1.2	1.4	1.2	5.8×10^5
1.11	1.2	1.16	1.2	6.0×10^5
1.0	1.2	1.04	1.2	6.2×10^5
1.2	1.2	1.2	1.3	1.5×10^4
1.2	1.3	1.2	1.4	1.6×10^4
1.2	1.4	1.2	2.0	1.3×10^5

Summary

– Some neutron star models are ‘closer’ together than others, but how can one quantify this?

– Introduce a space, which comes equipped with a natural distance measure, on which points represent different neutron star models. One can calculate distances and geodesics on this space, which then are (likely?) related to the closeness of models: The geometric structure captures the feature automatically (e.g. for Tolman VII stars).

– Future and ongoing work: non-Kerr backgrounds and black holes, applying to this a large data set of equations of state.

Kerr metric and smoking-guns

General Relativity

In general relativity, the Kerr metric represents the unique geometry surrounding an isolated, rotating black hole in vacuum.

– A detection of non-Kerr features would then imply a 'smoking-gun' for modified gravity: non-Einstein hairs or other parameters suggest a breakdown of general relativity in the strong-field regime.

– Such tests include: X-ray reflection spectroscopy, direct imaging with black hole shadows (cf. M87), and gravitational wave (GW) data analysis. Thus far, all results consistent with Kerr (but others too; Konoplya & Zhidenko 2016)