Make Nature Natural Again

Part 1: criticise all the rest

SUSY, you are fired;

Repeal standard naturalness;

Ban mass from theory.

Part 2: good crazy alt-phys

Dynamical generation of M_h, $M_{\rm Pl}$;

Infinite Energy;

Agravity, Ghosts; Inflation.

Alessandro Strumia
Pisa U. & INFN & CERN

CosmoVia, 19/5/2017

European Commission
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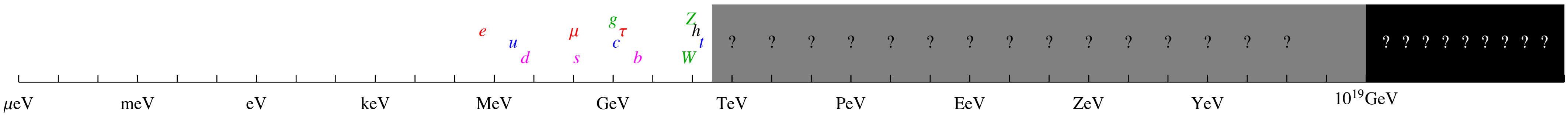
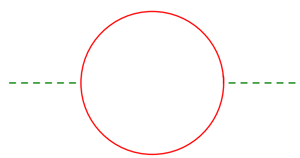
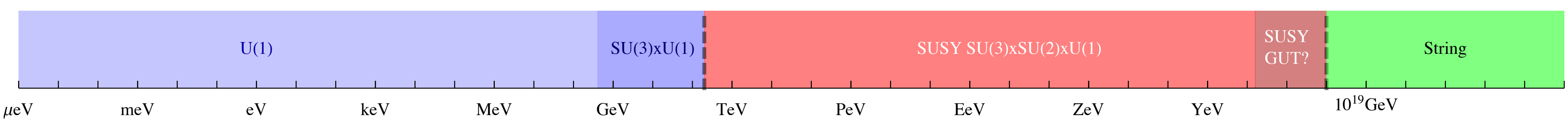
Mass scales in nature



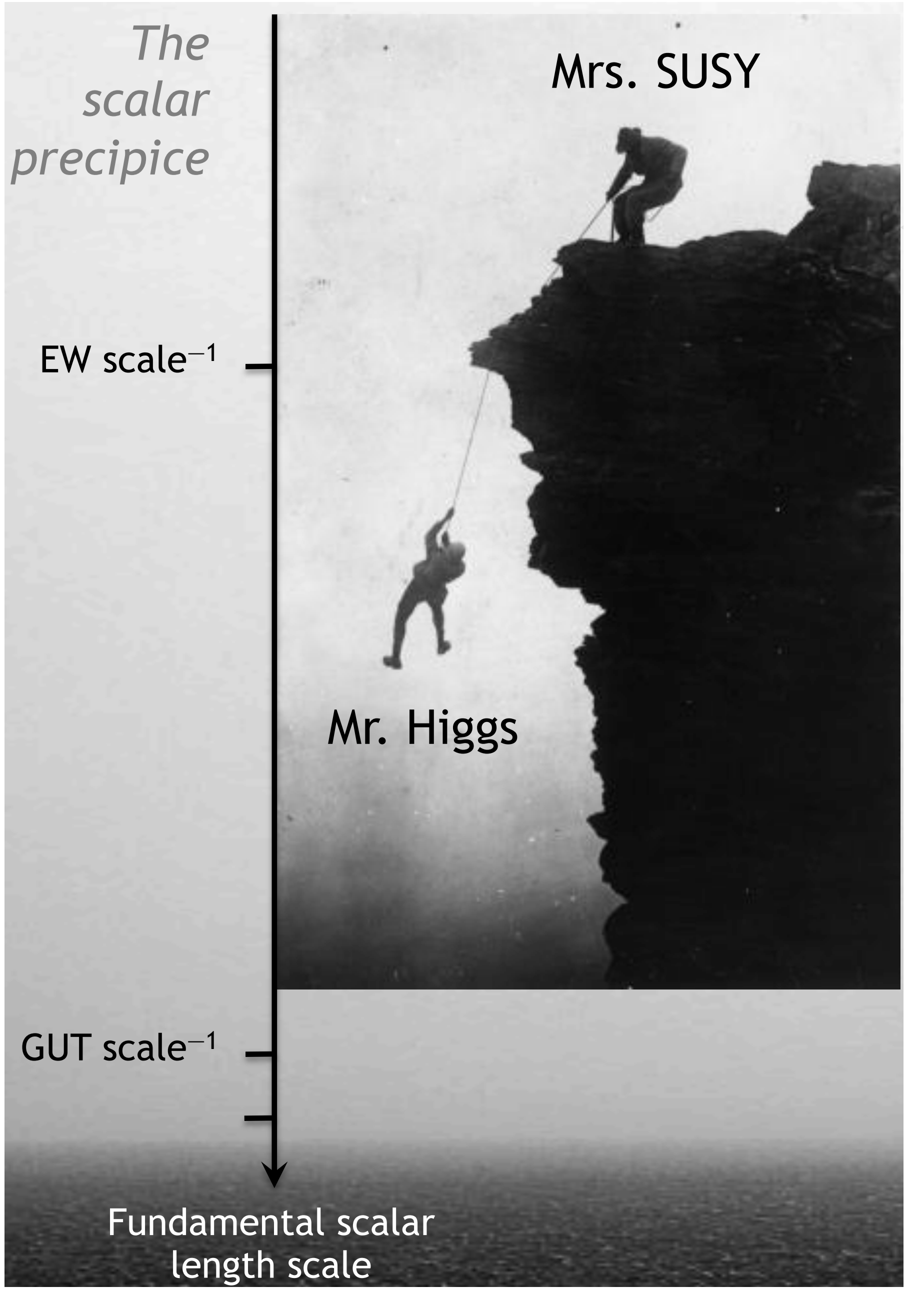
The SM explains part of the mess: $\Lambda_{\mathrm{QCD}} \sim M_{\mathrm{Pl}} e^{-2\pi/7\alpha_3}$ and $M_f = g_f \langle h \rangle$. But $M_h \ll M_{\mathrm{Pl}}$ not understood and apparently destabilized by quantum corrections:

$$\delta M_h^2 =$$  $$\sim g_{\mathrm{SM}}^2 \Lambda^2 \xrightarrow{?} g_{\mathrm{SM}}^2 M_{\mathrm{SUSY}}^2$$

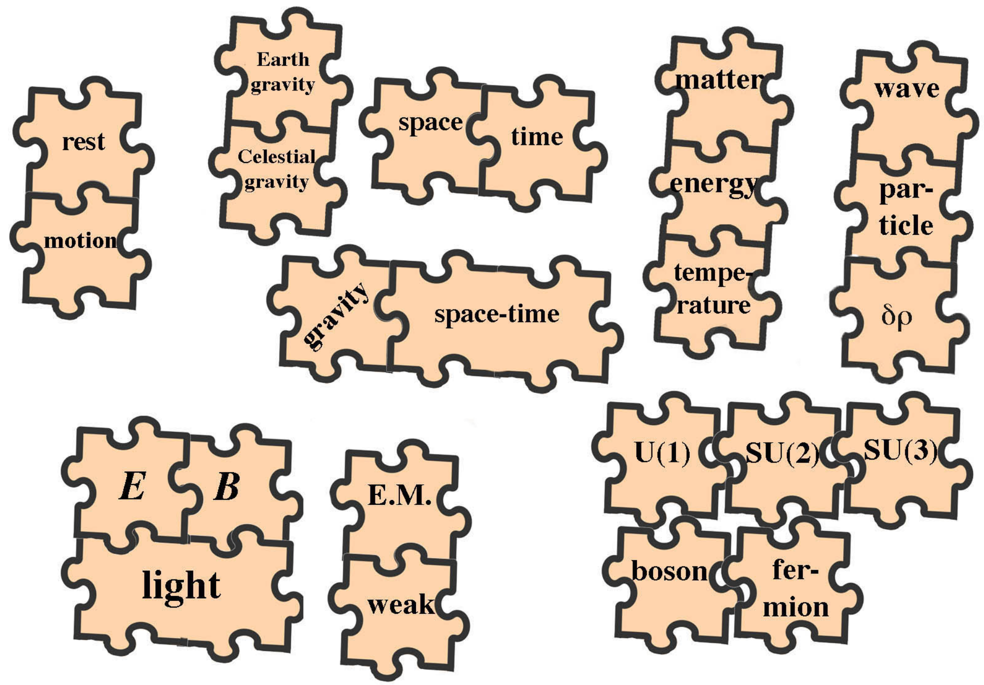
Dominant theory has two scales, the **string** scale and the EW/**SUSY** scale:



The establishment wants SUSY

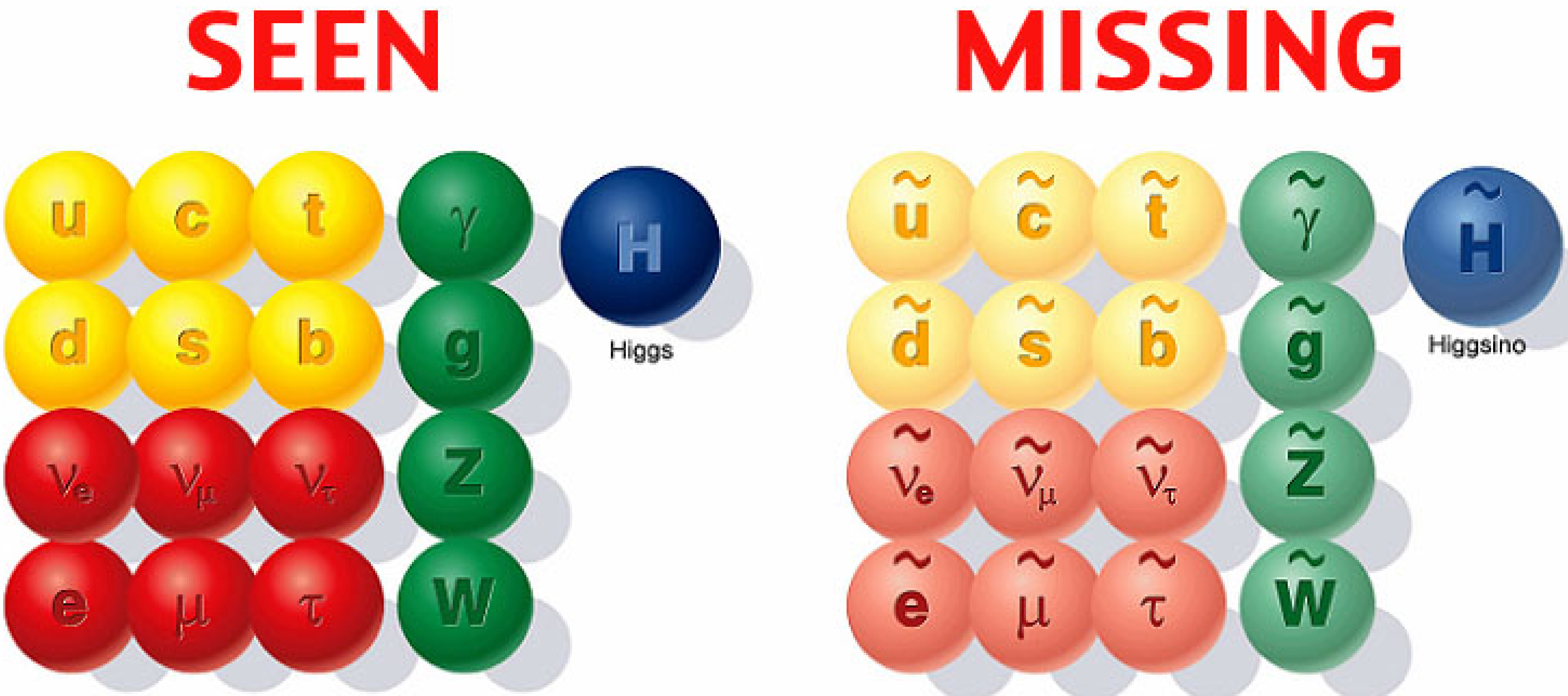


- ⋆ SUSY stabilizes Higgs: the weak scale is the scale of SUSY breaking.
- ⋆ SUSY extends Lorentz, allows spin 3/2.
- ⋆ SUSY unifies fermions with bosons.
- ⋆ SUSY unifies gauge couplings.
- ⋆ SUSY gives DM aka 'neutralino'.
- ⋆ SUSY is predicted by super-strings.



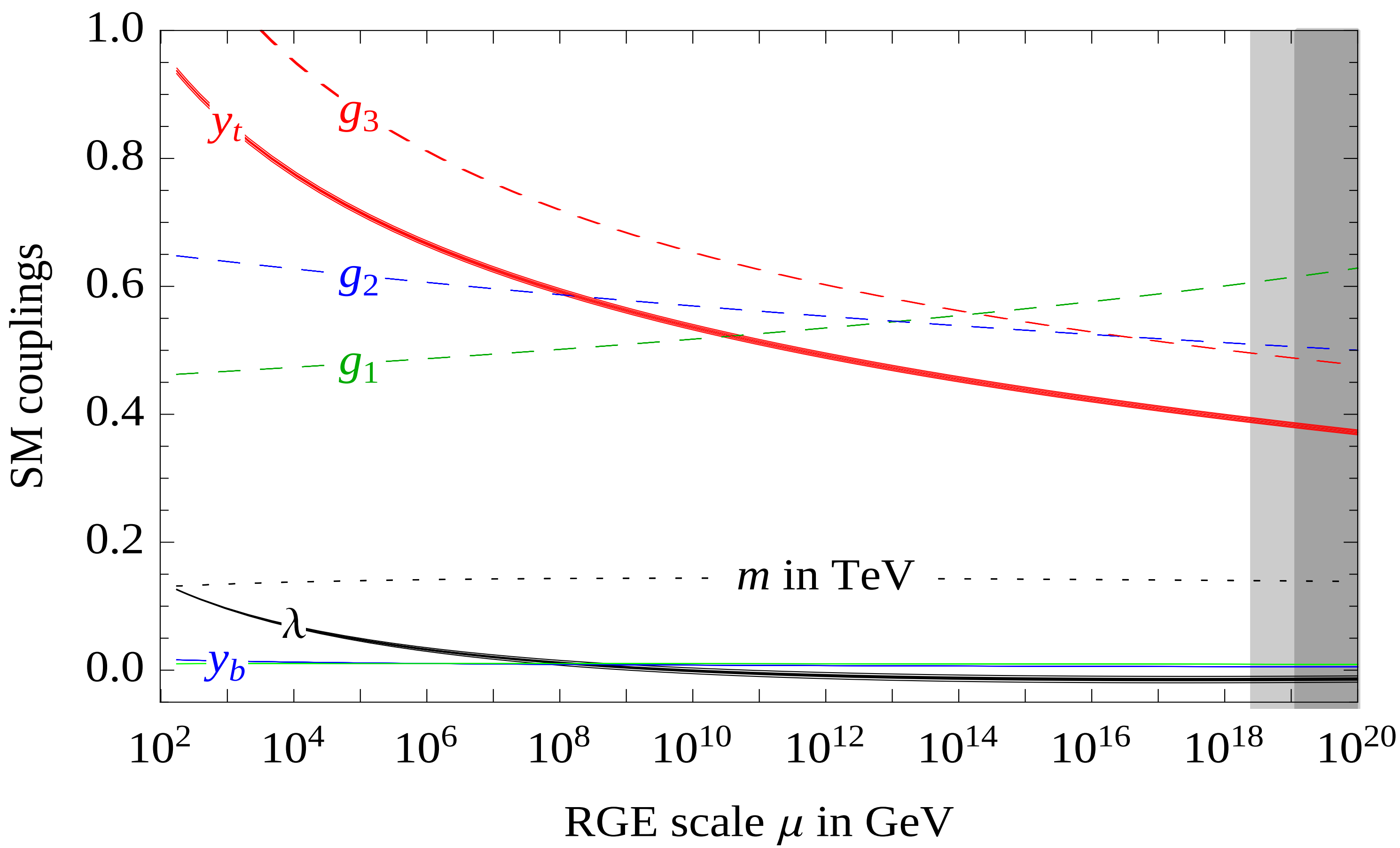
- ⋆ Worry: too many sparticles at LHC?

LHC inverse problem solved



News from the fronteer

No new physics at LHC. For the measured M_h, M_t the Standardissimo Model can be extrapolated up to $M_{\rm Pl}$ and above.



λ and its β-function nearly vanish around $M_{\rm Pl}$

Naturalness in trouble

SUSY was the best solution to a bigger issue: most theorists believe that

"light fundamental scalars must be accompanied by new physics that protects their lightness from quadratically divergent corrections"

But LHC observed the opposite: the Higgs and no new physics

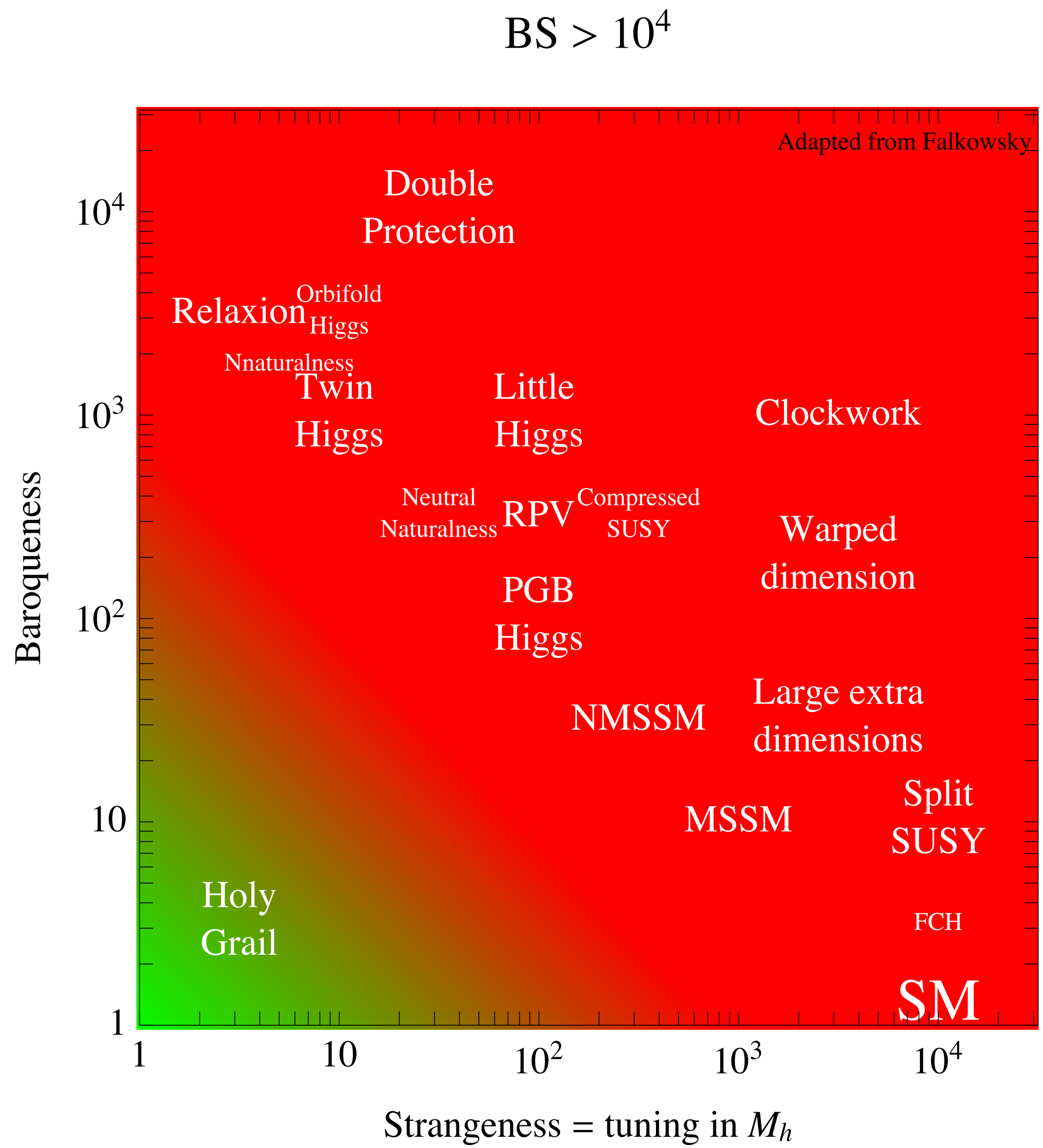
Confirmed at Moriond 2017. So many boring SM victories that the situation is interesting. All natural extensions of the SM in trouble: SUSY, extra diminesions, technicolor, composite Higgs...

These models no longer can be natural: $\delta M_h^2 \gtrsim 100 M_h^2$

Reaction 1: ad hoc ideas?

Add more smarter new physics to explain why we see nothing: compressed SUSY, RPV SUSY, 'neutral naturalness'... cosmological history that selects a small M_h.

Must be tried: no stone unturned. Looks like therapeutic obstinacy:



2: anthropic selection in a multiverse?

The cosmological constant $V \sim 10^{-120} M_{\rm Pl}^4$ is one more unnaturally small mystery. No natural theory known. Weinberg: anthropic selection in a multiverse.

Anthropics explains $M_h \ll M_{\rm Pl}$ too?

- Needed to have systems made of many particles.
- Chemistry exists thanks to $y_d v \approx \alpha_{\rm em} \Lambda_{\rm QCD}$.

But natural solutions exist, difficult to argue that multiverse avoids them.
Even if we live in a multiverse, natural anthropic theories would be more likely:

- SM with a smaller y or $M_{\rm Pl}$;
- a QED+QCD alternative without a Higgs;
- weak scale SUSY.

Keep searching alternatives to anthropic nirvana

Subtle is the Lord

What is going on? We are confused but nature is surely following some logic

The goal of this talk is presenting an alternative: a renormalizable theory valid above $M_{\rm Pl}$ such that M_h is naturally smaller than $M_{\rm Pl}$ without new physics at the weak scale. It naturally gives inflation and a beautiful anti-graviton ghost.

Reconsidering naturalness

Make Nature Natural Again

If nature looks unnatural, maybe we misunderstood what naturalness means.

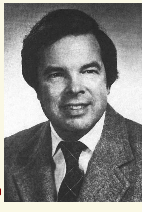
Power divergences and regulators are suggested by QFT equations. The æther was suggested by Maxwell equations. But power divergences are unphysical. Maybe we are again over-interpreting, adding realism to quantum mechanics.

Maybe there are no regulators: a SM-like theory holds up to infinite energy.

[Caution: this is when rotten tomatoes start to fly]

Wilson proposed usual naturalness attributing physical meaning to momentum shells of loop integrals, used in the 'averaged action'. Ipse undixit:

"The claim was that it would be unnatural for such particles to have masses small enough to be detectable soon. But this claim makes no sense".



Kenneth G. Wilson, hep-lat/0412043

Physical Naturalness

Demand that physical corrections only satisfy naturalness:

$$M_h \gtrsim \delta M_h \sim \begin{cases} g_{\rm SM} \Lambda_{\rm UV} & \text{Usual naturalness} \\ g_{\rm extra} M_{\rm extra} & \text{Physical naturalness} \end{cases}$$

The SM satisfies Physical Naturalness, for the measured $M_h \approx M_t$

This would be ruined by new heavy particles too coupled to the SM.

Unlike in the other scenarios, high-scale model building is very constrained.

Imagine there is no GUT. No flavour models too. Above us only sky.

Data demand some new physics: DM, neutrino masses, maybe axions...

Can this be added compatibly with Physical Naturalness?

Physical Naturalness and new physics

Neutrino mass models add extra particles with mass M

$$M \lesssim \left\{ \begin{array}{ll} 0.7\ 10^7\,\text{GeV} \times \sqrt[3]{\Delta} & \text{type I see-saw model,} \\ 200\,\text{GeV} \times \sqrt{\Delta} & \text{type II see-saw model,} \\ 940\,\text{GeV} \times \sqrt{\Delta} & \text{type III see-saw model.} \end{array} \right.$$

Leptogenesis is compatible with PhysNat only in type I.

Axion and LHC usually are like fish and bicycle because $f_a \gtrsim 10^9\,\text{GeV}$. Axion models can satisfy PN, e.g. KSVZ models employ heavy quarks with mass M

$$M \lesssim \sqrt{\Delta} \times \left\{ \begin{array}{ll} 0.74\,\text{TeV} & \text{if } \Psi = Q \oplus \bar{Q} \\ 4.5\,\text{TeV} & \text{if } \Psi = U \oplus \bar{U} \\ 9.1\,\text{TeV} & \text{if } \Psi = D \oplus \bar{D} \end{array} \right.$$

Inflation: flatness implies small couplings.

Dark Matter: below about a TeV if weakly coupled.

DM with weak gauge interactions

Consider a Minimal Dark Matter n-plet. 2-loop quantum corrections to M_h^2:

$$\delta M_h^2 = \frac{cnM^2}{(4\pi)^4}\left(\frac{n^2-1}{4}g_2^4 + Y^2 g_Y^4\right) \times \begin{cases} 6\ln\frac{M^2}{\Lambda^2} - 1 & \text{for fermion DM} \\ \frac{3}{2}\ln^2\frac{M^2}{\Lambda\mu^2} + 2\ln\frac{M^2}{\Lambda^2} + \frac{7}{2} & \text{for scalar DM} \end{cases}$$

Quantum numbers $SU(2)_L$	$U(1)_Y$	Spin	DM could decay into	DM mass in TeV	$m_{DM^\pm} - m_{DM}$ in MeV	Physical naturalness bound in TeV, $\Lambda \sim M_{Pl}$	σ_{SI} in $10^{-46}\,cm^2$
2	1/2	0	EL	0.54	350	$0.4 \times \sqrt{\Delta}$	$(2.3 \pm 0.3)\,10^{-2}$
2	1/2	1/2	EH	1.1	341	$1.9 \times \sqrt{\Delta}$	$(2.5 \pm 0.8)\,10^{-2}$
3	0	0	HH^*	2.5	166	$0.22 \times \sqrt{\Delta}$	0.60 ± 0.04
3	0	1/2	LH	2.7	166	$1.0 \times \sqrt{\Delta}$	0.60 ± 0.04
3	1	0	HH, LL	1.6+	540	$0.22 \times \sqrt{\Delta}$	0.06 ± 0.02
3	1	1/2	LH	1.9+	526	$1.0 \times \sqrt{\Delta}$	0.06 ± 0.02
4	1/2	0	HHH^*	2.4+	353	$0.14 \times \sqrt{\Delta}$	1.7 ± 0.1
4	1/2	1/2	(LHH^*)	2.4+	347	$0.6 \times \sqrt{\Delta}$	1.7 ± 0.1
4	3/2	0	HHH	2.9+	729	$0.14 \times \sqrt{\Delta}$	0.08 ± 0.04
4	3/2	1/2	(LHH)	2.6+	712	$0.6 \times \sqrt{\Delta}$	0.08 ± 0.04
5	0	0	(HHH^*H^*)	9.4	166	$0.10 \times \sqrt{\Delta}$	5.4 ± 0.4
5	0	1/2	stable	11.5	166	$0.4 \times \sqrt{\Delta}$	5.4 ± 0.4

A new principle: nature has no scale

Physical Naturalness is phenomenologically viable, what about its theory?

A naive effective field theory suffers of the hierarchy problem:

$$\mathscr{L} \sim \Lambda^4 + \Lambda^2 H^2 + \mathscr{L}_4 + \frac{H^6}{\Lambda^2} + \cdots$$

Nature is singling out $\mathscr{L}_4$. Why?

Principle: "**Nature has no fundamental scales Λ**".

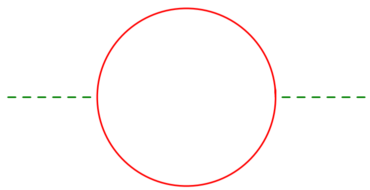
Then, the fundamental QFT is described by $\mathscr{L}_4$: only dimensionless couplings.

Power divergences have mass dimension. So they must vanish if there are no masses: $\int dE\, E = 0$. Anything different is dimensionally wrong.

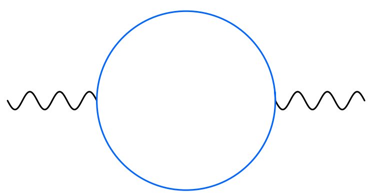
The scale anomaly

Is all this useless because quantum corrections break scale invariance?
- The chiral anomaly does not make fermions massive.
- The scale anomaly does not make scalars massive.

The one loop correction to a scalar mass^2 is quadratically divergent:

$$\Pi(0) =$$  $$= -4y^2 \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + m^2}{(k^2 - m^2)^2}$$

The same happens for the photon:

$$\Pi_{\mu\nu}(0) =$$  $$= -4e^2 \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \left(\frac{2k_\mu k_\nu}{(k^2 - m^2)^2} - \frac{\eta_{\mu\nu}}{k^2 - m^2} \right)$$

And for the graviton $\Pi_{\mu\nu\alpha\beta}(0)$. A physical cut-off that respects gauge invariance and that breaks scale invariance (such as strings) can keep $M_\gamma = 0$, while $M_h \sim \Lambda$. But M_γ and M_h have the same fate in a theory with no cut-off.

Can quantum corrections generate $M_h, M_{\rm Pl}$?

Yes, if dynamics generates vevs or condensates. 1) Models for M_h; 2) for $M_{\rm Pl}$.

1) What is the weak scale?

$M_h \sim g_{\text{extra}} M_{\text{extra}}$ where g_{extra} can be $\ll g_{\text{SM}}$, so M_{extra} can be $\gg M_h$

Physical naturalness does not imply new physics at the weak scale

- Could be generated from nothing by weak-scale dynamics.
  - Another gauge group might become strong around 1 TeV.
  - The quartic of another scalar might run negative around 1 TeV.

- Could be generated from nothing by heavier dynamics.
  - See-saw, axions, gravity...

Weakly coupled models for the weak scale

The Coleman-Weinberg mechanism can dynamically generate the weak scale

Model :

$G_{\text{SM}} \otimes \text{SU}(2)_X$ with one extra scalar S, doublet under $\text{SU}(2)_X$ and potential

$$V = \lambda_H |H|^4 - \lambda_{HS} |HS|^2 + \lambda_S |S|^4.$$

1) **Dynamically generates** the weak scale and weak scale DM
2) **Preserves** the successful automatic features of the SM: B, L...
3) **Gets DM stability** as one extra automatic feature.

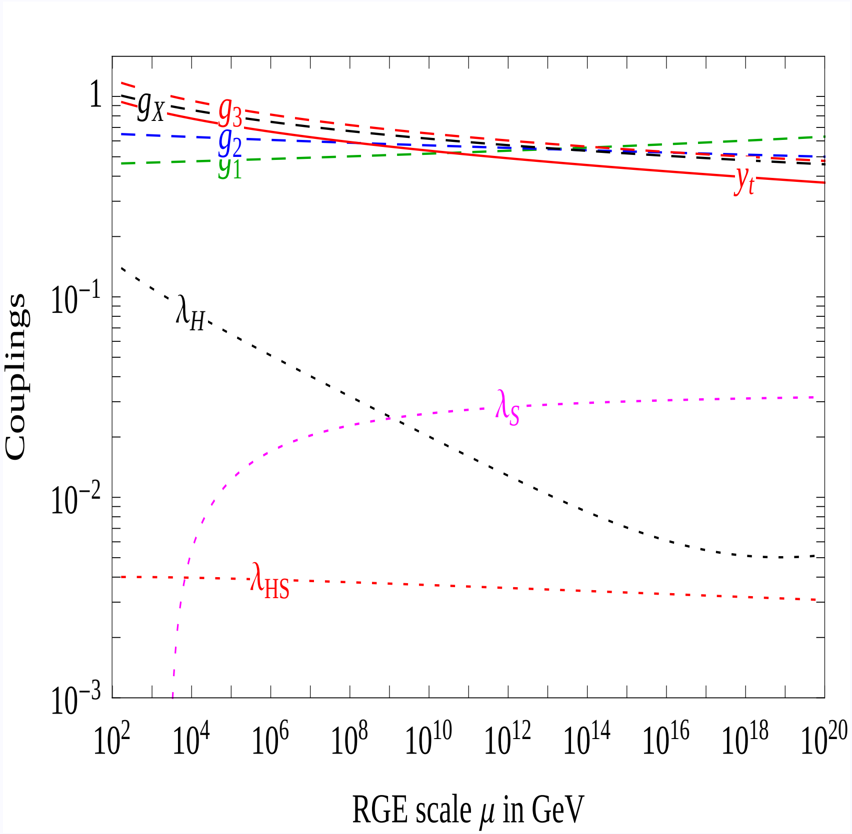
Weakly coupled SU(2) model

1) λ_S runs negative at low energy:

$$\lambda_S \simeq \beta_{\lambda_S} \ln \frac{s}{s_*} \qquad \text{with} \qquad \beta_{\lambda_S} \simeq \frac{9 g_X^4}{8(4\pi)^2}$$

$$S(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \text{[redacted]} \\ w + s(x) \end{pmatrix} \qquad w \simeq s_* e^{-1/4}$$

$$H(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \qquad v \simeq w \sqrt{\frac{\lambda_{HS}}{2\lambda_H}}$$



2) No new Yukawas.

3) SU(2)$_X$ vectors get mass $M_X = \frac{1}{2} g_X w$ and are automatically stable.

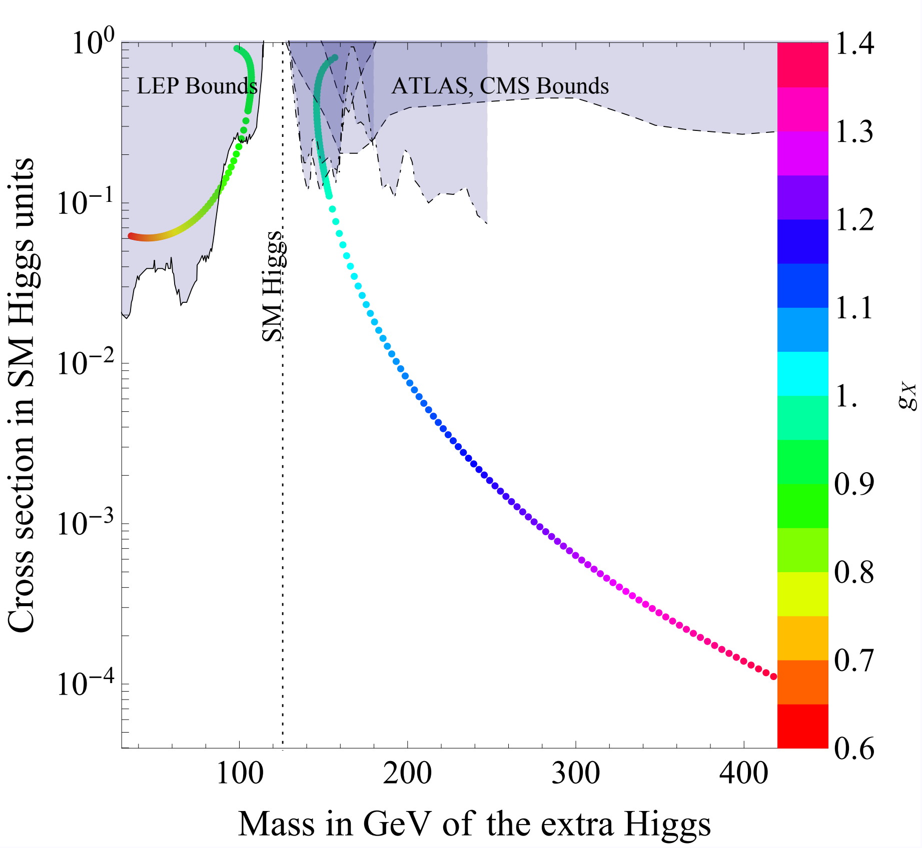
4) Bonus: threshold effect stabilises $\lambda_H = \lambda + \lambda_{HS}^2/\beta_{\lambda_S}$.

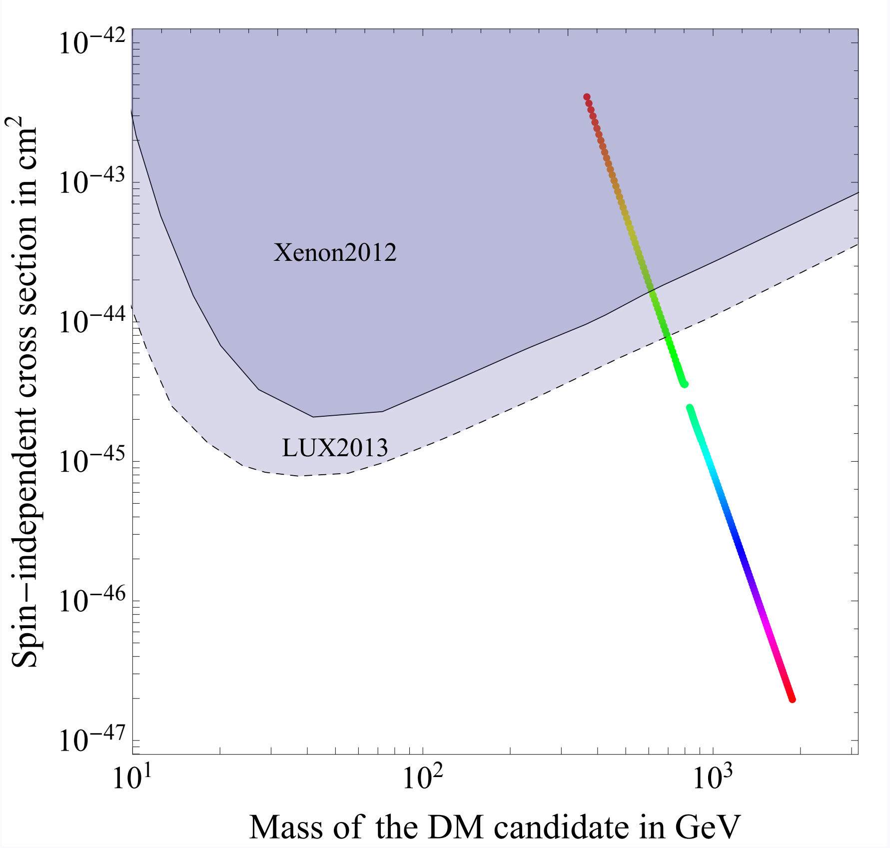
Experimental implications

1) New scalar s: like another h with suppressed couplings; $s \to hh$ if $M_s > 2M_h$.
2) Dark Matter coupled to s, h. Assuming that DM is a thermal relict

$$\sigma v_{\rm ann} + \frac{1}{2}\sigma v_{\rm semi-ann} = \frac{11 g_X^2}{1728\pi w^2} + \frac{g_X^2}{64\pi w^2} \approx 2.2\times 10^{26}\,\frac{\rm cm^3}{\rm s}$$

fixes $g_X = w/2\,\text{TeV}$, so all is predicted in terms of one parameter e.g. g_X:





Dark/EW phase transition is 1st order: gravitational waves, axiogenesis?

The weak scale from strong dynamics

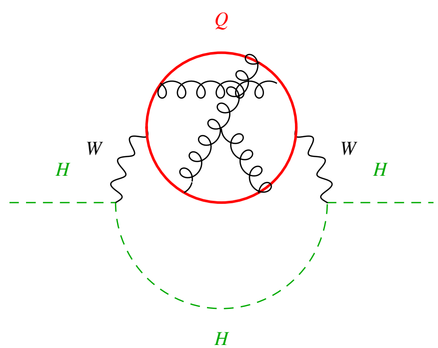
Model:

$G_{\text{SM}} \otimes \text{SU}(N)$ with one extra fermion in the $(0_Y, 3_L, 1_c, N \oplus \bar{N})$. $V = \lambda_H |H|^4$

No extra scalars, no masses: as many parameters as the SM!

The weak scale from strong dynamics

New QCD-like dynamics becomes strong at $\Lambda \sim$ few TeV inducing

$m_h^2 =$ 

$= \dfrac{9g_2^4}{4(4\pi)^2}\int dQ^2 \Pi_W(-Q^2)$

The W propagator contains strong dynamics. Dispersion relations proof $m_h^2 < 0$

$$\frac{\partial \Pi_W}{\partial \Lambda_{\text{TC}}^2} = -\frac{q^2}{\Lambda_{\text{TC}}^2}\frac{\partial \Pi_W}{\partial q^2}, \qquad \frac{\partial \Pi_W(q^2)}{\partial q^2} = \frac{1}{\pi}\int_0^\infty ds\, \frac{\overbrace{\text{Im}\,\Pi_W(s)}^{\sim -\sigma < 0}}{(s-q^2)^2} < 0$$

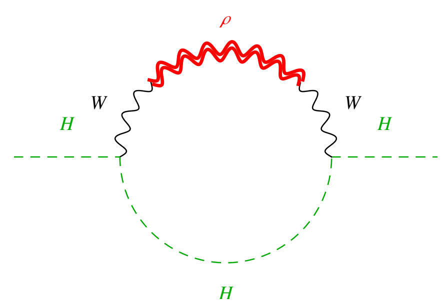
The weak scale from strong dynamics

Ignoring power divergences m_h^2 is UV-finite: use Operator Product Expansion

$$\Pi_W(q^2) \stackrel{q^2 \gg \Lambda^2}{\simeq} \underbrace{c_1(q^2)}_{\text{dimensionless}} + \underbrace{c_3(q^2)}_{-C/q^4} \underbrace{\langle 0|\frac{\alpha_{\text{TC}}}{4\pi}\mathcal{G}^{A2}_{\mu\nu}|0\rangle}_{\text{positive}} + \cdots$$

Vector Meson Dominance estimates $\Pi_W(q^2) = m_\rho^2/g_\rho^2(q^2 - m_\rho^2 + i\epsilon)$

$$m_h^2 \sim$$



$$\sim -\frac{g_2^4 m_\rho^2}{(4\pi)^2 g_\rho^2}$$

All new physics univocally predicted: $m_\rho \sim 20\,\text{TeV}$, 'baryons' at $m_B \sim 50\,\text{TeV}$. Lighter 'pions' in the $3 \otimes 3 - 1 = 3 \oplus 5$ of $\text{SU}(2)_L$ at $m_{\pi_n} \approx \frac{g_2 m_\rho}{4\pi}\sqrt{\frac{3}{4}(n^2-1)} \sim 2\,\text{TeV}$. π_5 decays via the anomaly $\pi_5 \to WW$.

Dark Matter from strong dynamics

The model has **two** accidentally stable composite DM candidates:

- **The lightest 'baryon'**, presumably subdominant:

$$\Omega_{\text{thermal}} \approx 0.1 \left(\frac{m_B}{200\,\text{TeV}}\right)^2$$

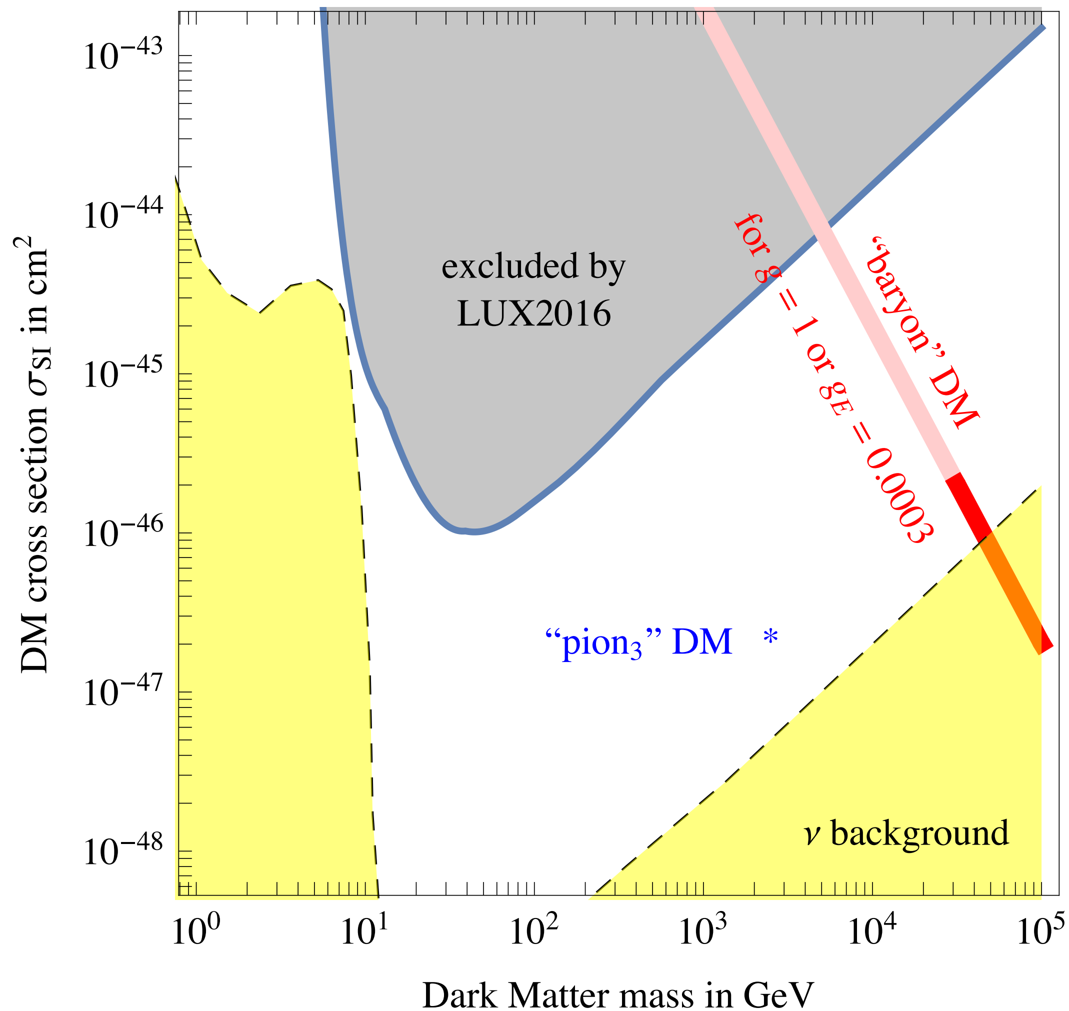
Characteristic magnetic dipole direct detection interaction.

- **The 'pion'** π_3. Thermal relic abundance predicted, ok for

$$m_{\pi_3} = 2.5\,\text{TeV}$$

Direct detection:

$$\sigma_{\text{SI}} \approx 0.2\ 10^{-46}\,\text{cm}^2.$$



Soft gravity

$$M_h \gtrsim \delta M_h \sim \begin{cases} g_{\rm SM}\Lambda_{\rm UV} & \text{Usual naturalness} \\ g_{\rm extra} M_{\rm extra} & \text{Physical naturalness} \end{cases}$$

The Einstein gravitational coupling grows with energy, blows up at $M_{\rm Pl}$

$$g_{\rm grav} \sim E/M_{\rm Pl}$$

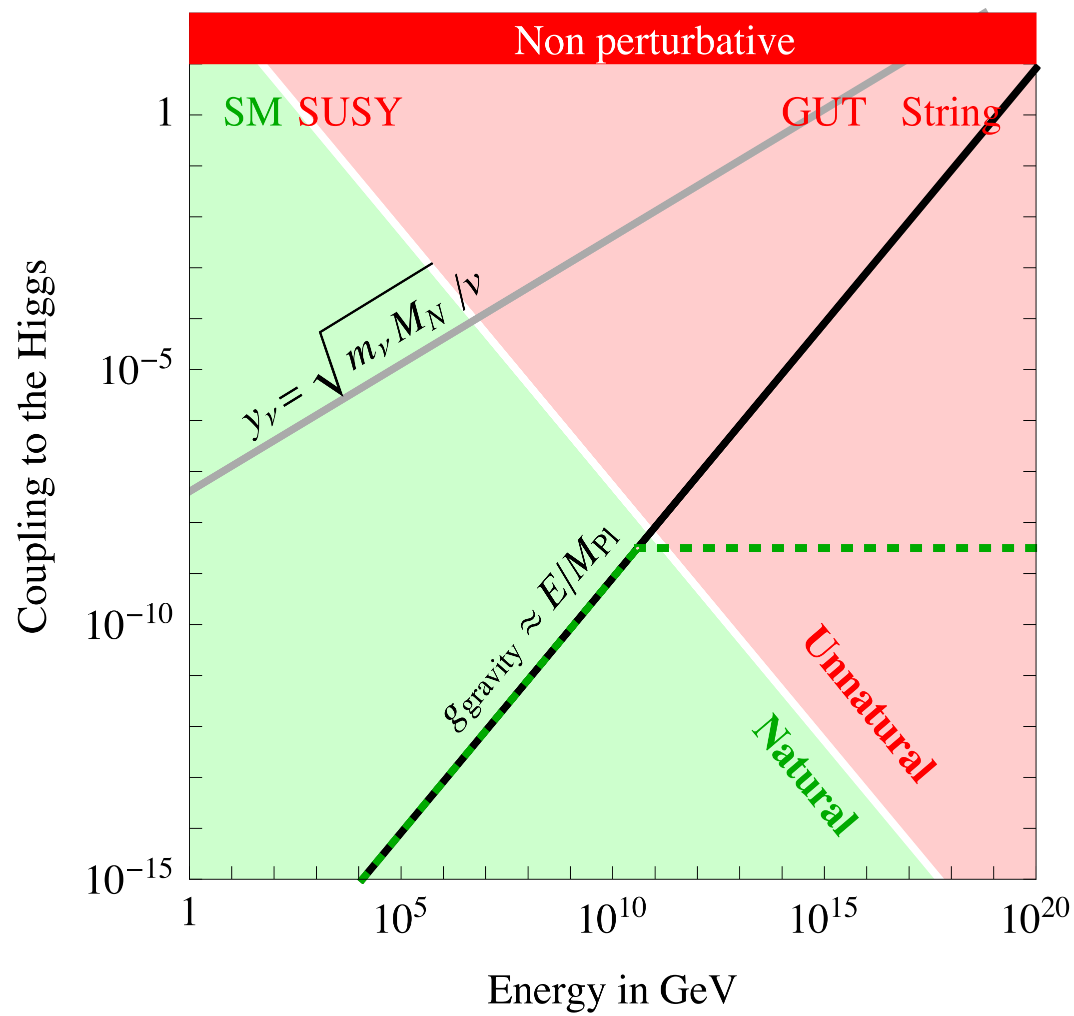
and couples everybody:

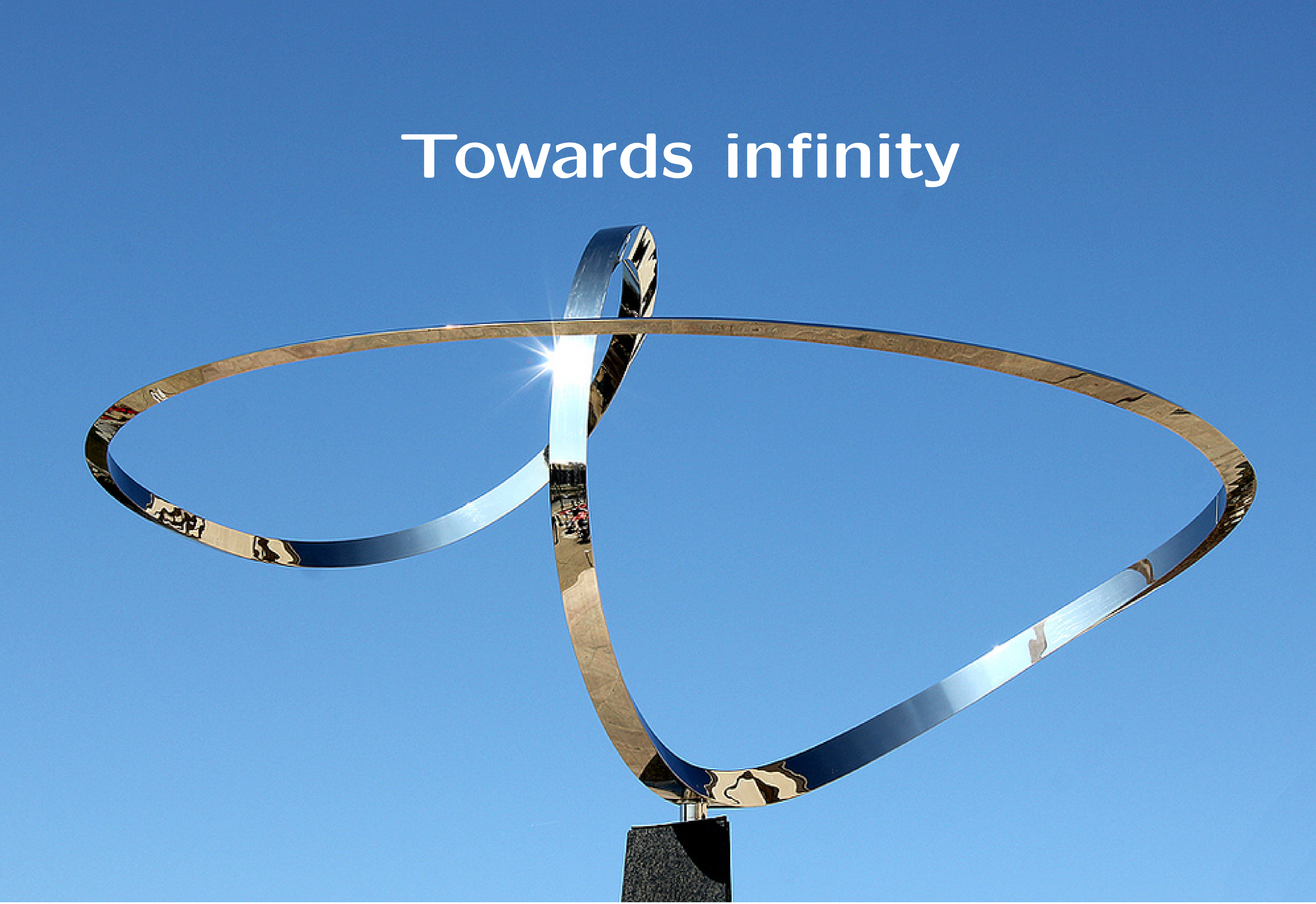
$$\delta M_h \sim g_{\rm grav} M_{\rm extra} \sim M^2_{\rm extra}/M_{\rm Pl}.$$

New physics must fix gravity when it is natural

$$g_{\rm grav} \lesssim 10^{-8}$$

$$M_{\rm extra} \lesssim 10^{12}\,\text{GeV}$$




Towards infinity

Motivation

If the theory has no cut-off Λ, it cannot give $\delta M_h^2 \sim \Lambda^2$

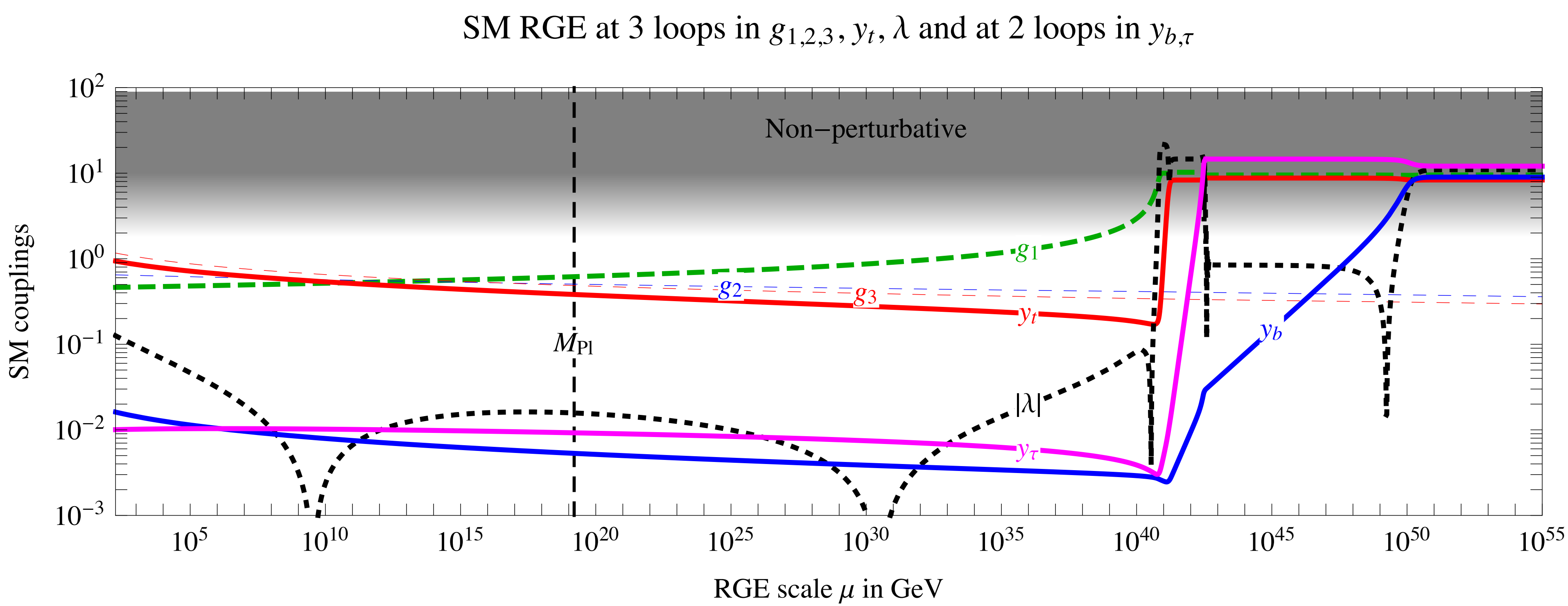
Models of soft gravity (agravity later) give RGE above $M_{\rm Pl}$.
We assume that the gravitational coupling is numerically small.
So RGE are dominated by the bigger QFT couplings: $g_{1,2,3}, y_t, \ldots$

Can the theory reach infinite energy?

Obstacle: Landau poles

Asymptotically safe Higgs?

In the SM, the abelian g_Y runs non-perturbative at $\Lambda \sim 10^{40}$ GeV.
Maybe the SM dies there? Maybe $g_Y, y_t, \lambda, \ldots$ enter into asymptotic safety?



Would this imply an unnatural $\delta M_h^2 \sim g_Y^2 \Lambda^2$? No such correction is present in a toy SM where couplings enter into perturbative Total Asymptotic Safety.

We don't know how to compute if the SM is TAS, so we explore TAF

Total Asymptotic Freedom?

Goal: compute if **all** couplings of a realistic QFT can run to 0 to $E = \infty$.

Naive attempt:

- solve the RGE for g, y, λ numerically
- up to infinite energy
- identify m-dimensional sub-spaces.

Result:

Analytic tools needed

TAF tools

Rewrite RGE in terms of $t = \ln \mu^2/(4\pi)^2$ and of $x_I = \{\tilde{g}_i, \tilde{y}_a, \tilde{\lambda}_m\}$ as

$$g_i^2(t) = \frac{\tilde{g}_i^2(t)}{t}, \qquad y_a^2(t) = \frac{\tilde{y}_a^2(t)}{t}, \qquad \lambda_m(t) = \frac{\tilde{\lambda}_m(t)}{t}.$$

Get

$$\frac{dx_I}{d\ln t} = V_I(x) = \begin{cases} \tilde{g}_i/2 + \beta_{g_i}(\tilde{g}), \\ \tilde{y}_a/2 + \beta_{y_a}(\tilde{g}, \tilde{y}), \\ \tilde{\lambda}_m + \beta_{\lambda_m}(\tilde{g}, \tilde{y}, \tilde{\lambda}). \end{cases}$$

Fixed-points $x_I(t) = x_\infty$ are determined by the algebraic equation $V_I(x_\infty) = 0$.

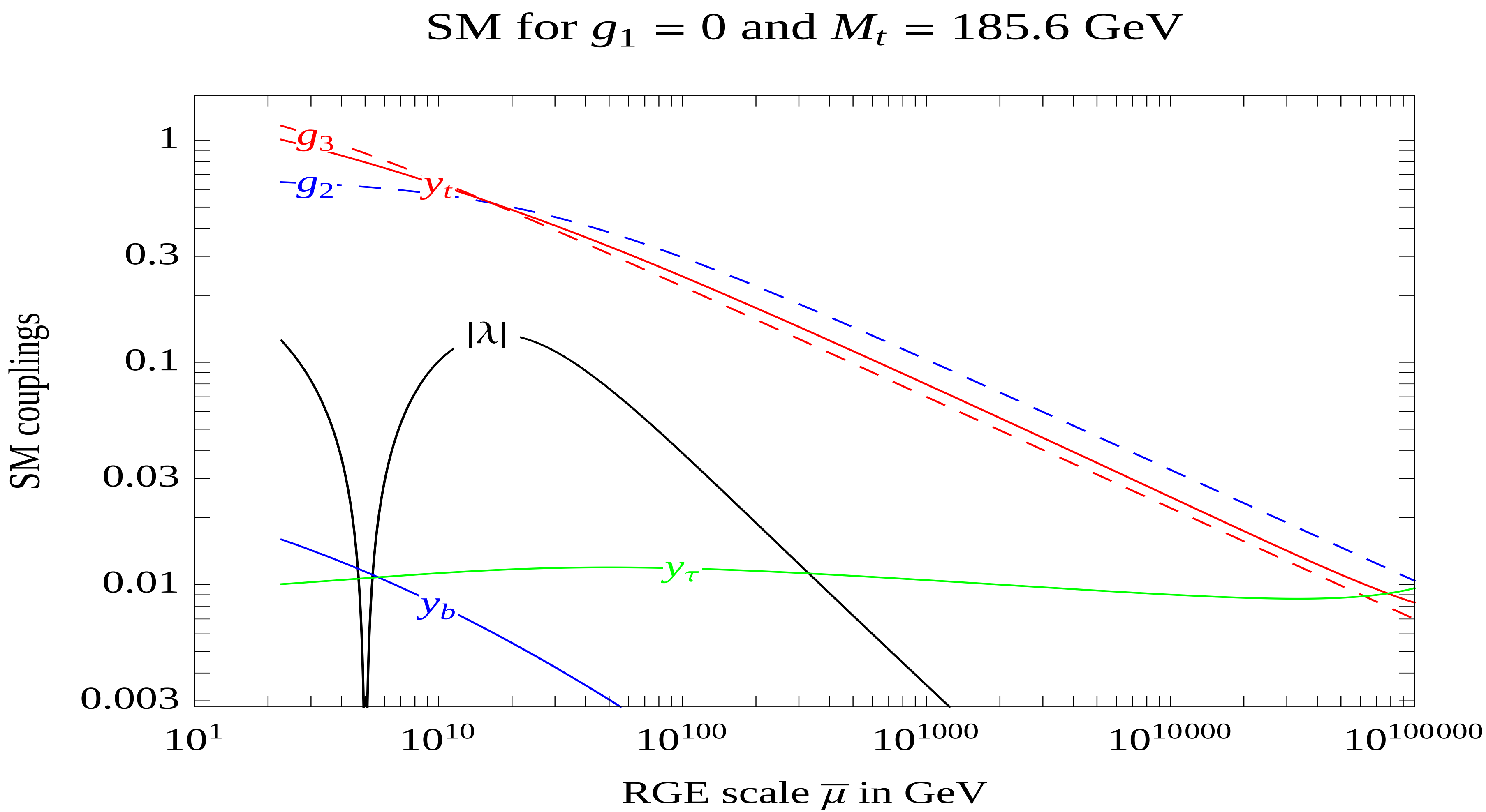
Linearize around each fixed-point:

$$V_I(x) \simeq \sum_J M_{IJ}(x_J - x_{J\infty}) \qquad \text{where} \qquad M_{IJ} = \left.\frac{\partial V_I}{\partial x_J}\right|_{x=x_\infty}$$

Negative eigenvalues of M are UV-attractive. Each positive eigenvalue implies a UV-repulsive direction: to reach the FP a coupling is univocally **predicted**.

SM up to infinite energy if $g_Y = 0$

Predictions: 1) $g_Y = 0$; in this limit 2) $y_t^2 \simeq 227/1197t$ i.e. $M_t = 186\,\text{GeV}$; 3) $y_{\tau,\nu} = 0$; 4) $\lambda \simeq (-143 \pm \sqrt{119402})/4788t$ i.e. $M_h \leq 163\,\text{GeV}$. Equality avoids $\lambda < 0$ at large energy, and too fast vacuum decay $\lambda < -1/12t$.



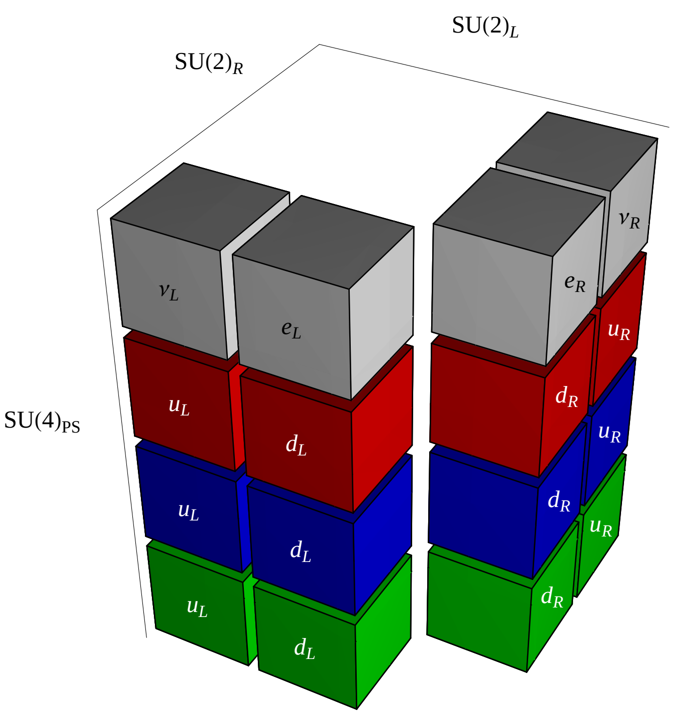
TAF extensions of the SM

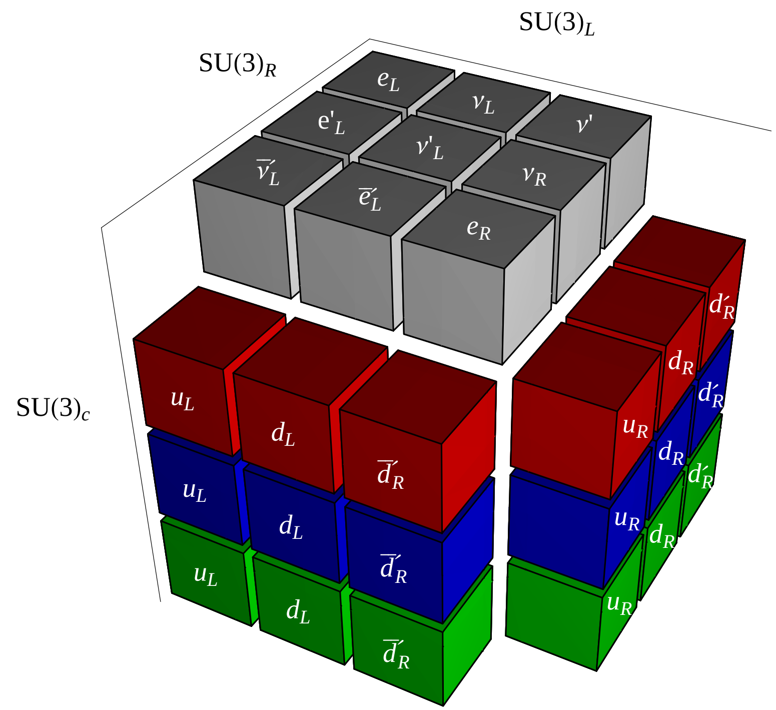
Can the SM be extended into a theory valid up to infinite energy?

Avoid Landau poles by making hypercharge non abelian.

We found realistic SU(5) TAF models. But GUTs are not compatible with finite naturalness, that demands a TAF extension at the weak scale. Making sense of $Y = T_{3R} + (B-L)/2$ needs $\mathrm{SU}(2)_R$. We see 2 possibilities:

$$\mathrm{SU}(4)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \qquad \text{and} \qquad \mathrm{SU}(3)_c \otimes \mathrm{SU}(3)_L \otimes \mathrm{SU}(3)_R$$





Generic signals of natural TAF

- A W_R boson and a Z'_{B-L}: $M_{W_R} \gtrsim 2.2\,\text{TeV}$, $M_{Z'_{B-L}} > 2.6_{333}, 3.8_{224}\,\text{TeV}$

$$\delta M_h^2 = -\frac{9 g_R^2 M_{W_R}^2}{(4\pi)^2} \ln\left(\frac{M_{W_R}^2}{\bar{\mu}^2}\right) \approx M_h^2 \left(\frac{M_{W_R}}{2.5\,\text{TeV}}\right)^2$$

- The Higgs $(2_L, \bar{2}_R)$ contains 2 doublets coupled to u and d: new flavour violations controlled by a right-handed CKM matrix.

$$M_H > \begin{cases} 18\,\text{TeV} & \text{if } V_R = V_{\text{CKM}} \\ 3\,\text{TeV} & \text{if } V_R^{ij} = V_{\text{CKM}}^{ij} \times \min(m_i, m_j)/\max(m_i, m_j) \text{ (natural texture)} \end{cases}$$

- A lighter singlet that mixes with the higgs if $G_{\text{TAF}} \to G_{\text{SM}}$ dynamically.

- And TAF is tough: we still have to find models where y, λ obey TAF

Pati-Salam

Fields	spin	generations	$SU(2)_L$	$SU(2)_R$	$SU(4)_{PS}$
$\psi_L = \begin{pmatrix} \nu_L & e_L \\ u_L & d_L \end{pmatrix}$	1/2	3	$\bar{2}$	1	4
$\psi_R = \begin{pmatrix} \nu_R & u_R \\ e_R & d_R \end{pmatrix}$	1/2	3	1	2	$\bar{4}$
ϕ_R	0	1	1	2	$\bar{4}$
$\phi = \begin{pmatrix} H_U^0 & H_D^+ \\ H_U^- & H_D^0 \end{pmatrix}$	0	2	2	$\bar{2}$	1
ψ	1/2	1, 2, 3	2	$\bar{2}$	1
Q_L	1/2	2	1	1	10
Q_R	1/2	2	1	1	$\overline{10}$
Σ	0	1	1	1	15

No extra chiral fermions. Two ways to get acceptable fermions masses:

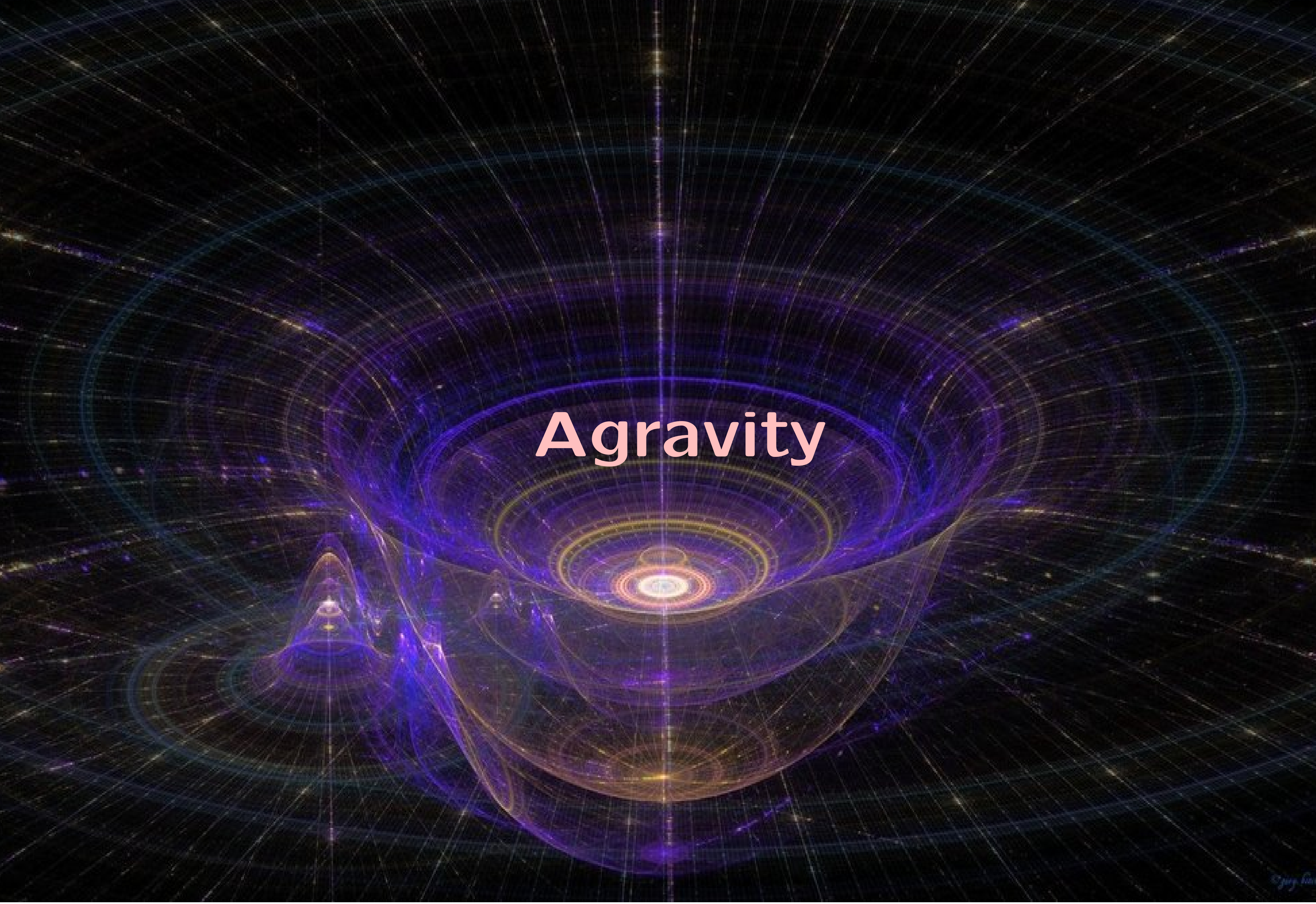
1) Foot: add ψ and ϕ_L: $-\mathscr{L}_Y = Y_N\,\psi_L\psi\phi_R + Y_L\,\psi\psi_R\phi_L + Y_U\psi_R\psi_L\phi + Y_D\,\psi_R\psi_L\phi^c$. Avoids ℓ_L/d_L unification so $M_{W'} > 8.8\,\text{TeV}$. No TAF found for the 24 quartics.

2) Volkas: add $Q_{L,R}$ getting d_R mixing. Strong flavor bounds $M_{W'} > 100\,\text{TeV}$ because of ℓ_L/d_L unification. **Unnatural**. TAF found adding Σ.

Trinification

Minimal weak-scale trinification model					
Matter fields	gen.s	spin	$\mathrm{SU}(3)_L$	$\mathrm{SU}(3)_R$	$\mathrm{SU}(3)_c$
$Q_R = \begin{pmatrix} u_R^1 & u_R^2 & u_R^3 \\ d_R^1 & d_R^2 & d_R^3 \\ d_R'^1 & d_R'^2 & d_R'^3 \end{pmatrix}$	3	1/2	1	3	$\bar{3}$
$Q_L = \begin{pmatrix} u_L^1 & d_L^1 & \bar{d}_R'^1 \\ u_L^2 & d_L^2 & \bar{d}_R'^1 \\ u_L^3 & d_L^3 & \bar{d}_R'^3 \end{pmatrix}$	3	1/2	$\bar{3}$	1	3
$L = \begin{pmatrix} \bar{\nu}_L' & e_L' & e_L \\ \bar{e}_L' & \nu_L' & \nu_L \\ e_R & \nu_R & \nu' \end{pmatrix}$	3	1/2	3	$\bar{3}$	1
$\langle H \rangle = \begin{pmatrix} v_u & 0 & 0 \\ 0 & v_d & v_L \\ 0 & V_R & V \end{pmatrix}$	3	0	3	$\bar{3}$	1

- Explains quantisation of Y. Needs $g_R = 2g_2 g_Y/\sqrt{3g_2^2 - g_Y^2} \approx 0.65 g_2$.
- No bad vectors: $V \approx$ few TeV allowed.
- Extra d', e', ν' fermions chiral under $\mathrm{SU}(3)^3$ get mass $\sim yV$ from Yukawas $y_Q\, Q_L Q_R H + \frac{1}{2} y_L^n LLH^*$. $3H$ are needed to make d', e', ν' naturally heavy.
- TAF solutions found for H_1, H_2 (20 quartics) and for H_1, H_2, H_3 (90 λ).


Agravity

What about gravity?

Does quantum gravity give $\delta M_h^2 \sim M_{\mathsf{Pl}}^2$ ruining Physical Naturalness?

Yes in string models, where lots of new coupled particles exists around M_{Pl}.

Maybe M_{Pl}^{-1} is just a small coupling and there are no new particles around M_{Pl}.

Adimensional gravity

Applying the adimensional principle to the SM plus gravity and a scalar S gives:

$$\mathscr{S} = \int d^4x \,\sqrt{|\det g|}\,\mathscr{L}$$

$$\mathscr{L} = \mathscr{L}_{\rm SM} + \frac{R^2}{3f_0^2} + \frac{R^2 - 3R_{\mu\nu}^2}{3f_2^2} + |D_\mu S|^2 - \xi_S|S|^2 R - \lambda_S|S|^4 + \lambda_{HS}|HS|^2$$

where f_0, f_2 are the adimensional 'gauge couplings' of gravity and $R \sim \partial_\mu\partial_\nu g_{\mu\nu}$.

Of course the theory is renormalizable, and indeed the graviton propagator is:

$$\frac{-i}{k^4}\left[2f_2^2 P^{(\rm spin\ 2)}_{\mu\nu\rho\sigma} - f_0^2 P^{(\rm spin\ 0)}_{\mu\nu\rho\sigma} + \text{gauge-fixing}\right].$$

The Planck scale should be generated dynamically as $\xi_S\langle S\rangle^2 = \bar{M}_{\rm Pl}^2/2$.

Then, the spin-0 part of $g_{\mu\nu}$ gets a mass $M_0 \sim f_0 M_{\rm Pl}$ and the spin 2 part splits into the usual graviton and an anti-graviton with mass $M_2 = f_2\bar{M}_{\rm Pl}/\sqrt{2}$ that acts as a Pauli-Villars in view its negative kinetic term [Stelle, 1977].

A ghost?

A ghost?

In presence of masses, ∂^4 can be decomposed as 2 fields with 2 derivatives:

$$\frac{1}{k^4} \to \frac{1}{k^4 - M_2^2 k^2} = \frac{1}{M_2^2}\left[\frac{1}{k^2} - \frac{1}{k^2 - M_2^2}\right]$$

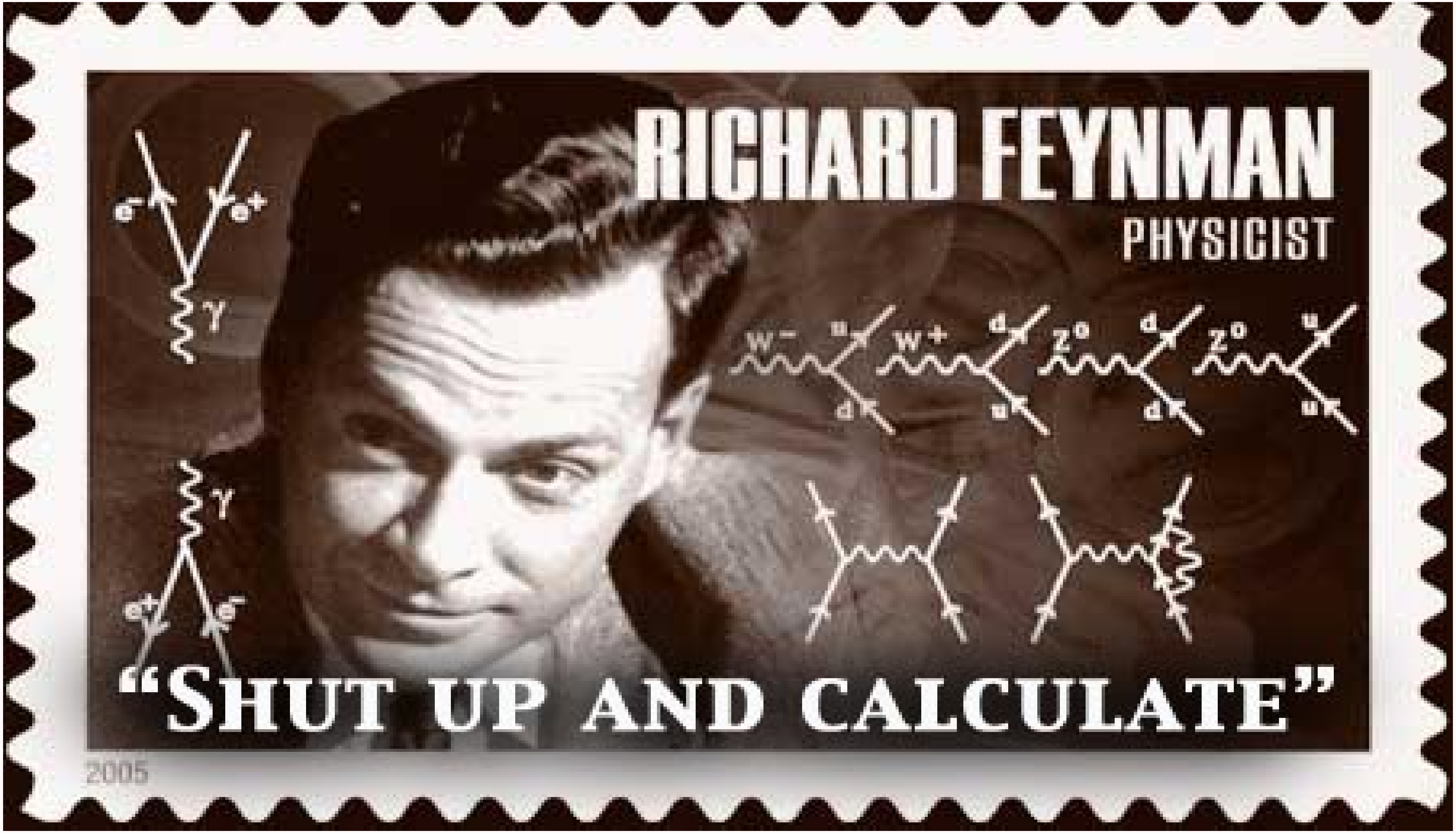
Ostrogradski showed in 1850 that higher derivatives are always bad:

$\partial^4 \Rightarrow$ unbounded **negative energy** $\Rightarrow$ the **classical** theory is dead.

Who cares, nature is quantum. ∂^4 can be quantized as:
i) negative energy, or as ii) **negative norm and positive energy**.
This is the $i\epsilon$ choice that makes agravity renormalizable.
A non-sense or just a slightly acausal unitary S matrix?

For the moment, let's ignore the issue and compute. Anti-particles teach us that sometimes we get the right equations before understanding their meaning.

A ghost?



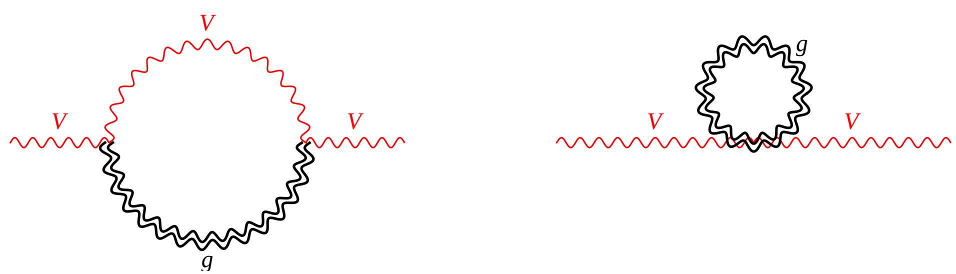
Quantum Agravity...

The quantum behaviour of a renormalizable theory is encoded in its RGE.
The unusual $1/k^4$ makes easy to get signs wrong. Literature is contradictory.

- f_2 is asymptotically free:

$$(4\pi)^2 \frac{df_2^2}{d\ln\mu} = -f_2^4 \left[\frac{133}{10} + \frac{N_V}{5} + \frac{N_f}{20} + \frac{N_s}{60}\right]$$

- Gravity does not affect running of gauge couplings: these two diagrams cancel



presumably because abelian g is undefined without charged particles.

- f_0 grows with energy

$$(4\pi)^2 \frac{df_0^2}{d\ln\mu} = \frac{5}{3} f_2^4 + 5 f_2^2 f_0^2 + \frac{5}{6} f_0^4 + \frac{f_0^4}{12} \sum_s (1+6\xi_s)^2$$

...Quantum Agravity

- Yukawa couplings get an extra multiplicative RGE correction:

$$(4\pi)^2\frac{dy_t}{d\ln\mu} = \frac{9}{2}y_t^3 - y_t(8g_3^2 - \frac{15}{8}f_2^2)$$

- Agravity makes quartics small at low energy:

$$(4\pi)^2\frac{d\lambda_H}{d\ln\mu} = \xi_H^2[5f_2^4 + f_0^4(1+6\xi_H)^2] - 6y_t^4 + \frac{9}{8}g_2^4 + \cdots$$

- Agravity creates a mixed quartic:

$$(4\pi)^2\frac{d\lambda_{HS}}{d\ln\mu} = \frac{\xi_H\xi_S}{2}[5f_2^4 + f_0^4(6\xi_S+1)(6\xi_H+1)] + \text{multiplicative}$$

- RGE for ξ

$$(4\pi)^2\frac{d\xi_H}{d\ln\mu} = -\frac{5}{3}\frac{f_2^4}{f_0^2}\xi_H + f_0^2\xi_H(6\xi_H+1)(\xi_H+\frac{2}{3}) + (6\xi_H+1)\left[2y_t^2 - \frac{3}{4}g_2^2 + \cdots\right]$$

f_0 at the denominator can be avoided by writing RGE in terms of ‘gravitational quartics’ f_0^2, $f_0^2(\xi_H+1/6)$ and $\lambda_H + \frac{3}{8}f_0^2(\xi_H+1/6)^2$.

Up to infinite energy

Agravity can flow to conformal gravity at infinite energy: $f_0 = \infty$, $\xi = -1/6$.

f_0 grows until the conformal mode σ of the agraviton, $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$, gets strongly self-coupled. Nevertheless, conformal and shift symmetries of its action

$$\int d^4x \sqrt{|\det g|}\frac{R^2}{6f_0^2} = \frac{6}{f_0^2}\int d^4x[\Box\sigma + (\partial\sigma)^2]^2$$

imply $\beta(f_0) \sim 1/f_0^2$ at $f_0 \gg 1$: **no Landau pole**. σ fluctuates wildly but decouples if $\xi \to -1/6$, becoming a Weyl gauge redundancy such that

$$\lim_{f_0\to\infty} \frac{df_2^2}{d\ln\mu} = -\frac{f_2^4}{(4\pi)^2}\left[\frac{199}{15} + \frac{N_V}{5} + \frac{N_f}{20} + \frac{N_s}{60}\right]$$

At multi-loop level anomalies break conformal gravity giving agravity

$$\lim_{f_0\to\infty} \frac{d}{d\ln\bar\mu}\frac{1}{f_0^2} = -\frac{665 g_2^6}{216(4\pi)^8} + \frac{728 g_3^6}{9(4\pi)^8} + \frac{416\lambda_H^5}{5(4\pi)^{12}} + \cdots$$

$$\lim_{f_0\to\infty} \frac{d}{d\ln\bar\mu}(\xi_H + \frac{1}{6}) = 48\frac{\lambda_H^4}{(4\pi)^8} + \cdots$$

Generation of M_{Pl}

Mechanisms that can generate dynamically the Planck scale:

Non-perturbative: Some coupling g runs non-perturbative at M_{Pl}
Perturbative: Some quartic λ_S runs negative at M_{Pl}

Non-perturbative models are easily built. Add to the SM:

- An extra gauge group G_{TG} that becomes strong at $\Lambda_{\text{TG}} \sim M_{\text{Pl}}$.
- No scalars and no fermions charged under both G_{SM} and G_{TG}.

The sign of M_{Pl}^2 seems to depend on the (uncomputable?) strong dynamics.
The cosmological constant tends to be $V \sim -M_{\text{Pl}}^4$; specific models avoid it (e.g. adding a fermion in the adjoint of G_{TG}).

Generation of $M_{\rm Pl}$: perturbative

Add a scalar Planckion s with quantum potential $V(s) \approx \frac{1}{4}\lambda_S(\bar\mu \sim s)s^4$.
Usually it gets a Coleman-Weinberg minimum when λ_S runs negative.
The gravitational coupling ξ_S makes the vacuum equation non-standard:

$$\frac{\partial V}{\partial s} - \frac{4V}{s} = 0 \qquad \text{i.e.} \qquad \frac{\partial V_E}{\partial s} = 0$$

where $V_E = V/(\xi_S s^2)^2 \sim \lambda_S(s)/\xi_S^2(s)$ is the Einstein-frame potential. The vev

$$\langle s\rangle = \bar M_{\rm Pl}/\sqrt{\xi_S}$$

needs a condition different from the usual Coleman-Weinberg:

$$\frac{\beta_{\lambda_S}(\bar\mu \sim \langle S\rangle)}{\lambda_S(\bar\mu \sim \langle S\rangle)} - 2\frac{\beta_{\xi_S}(\bar\mu \sim \langle S\rangle)}{\xi_S(\bar\mu \sim \langle S\rangle)} = 0$$

The cosmological constant vanishes if

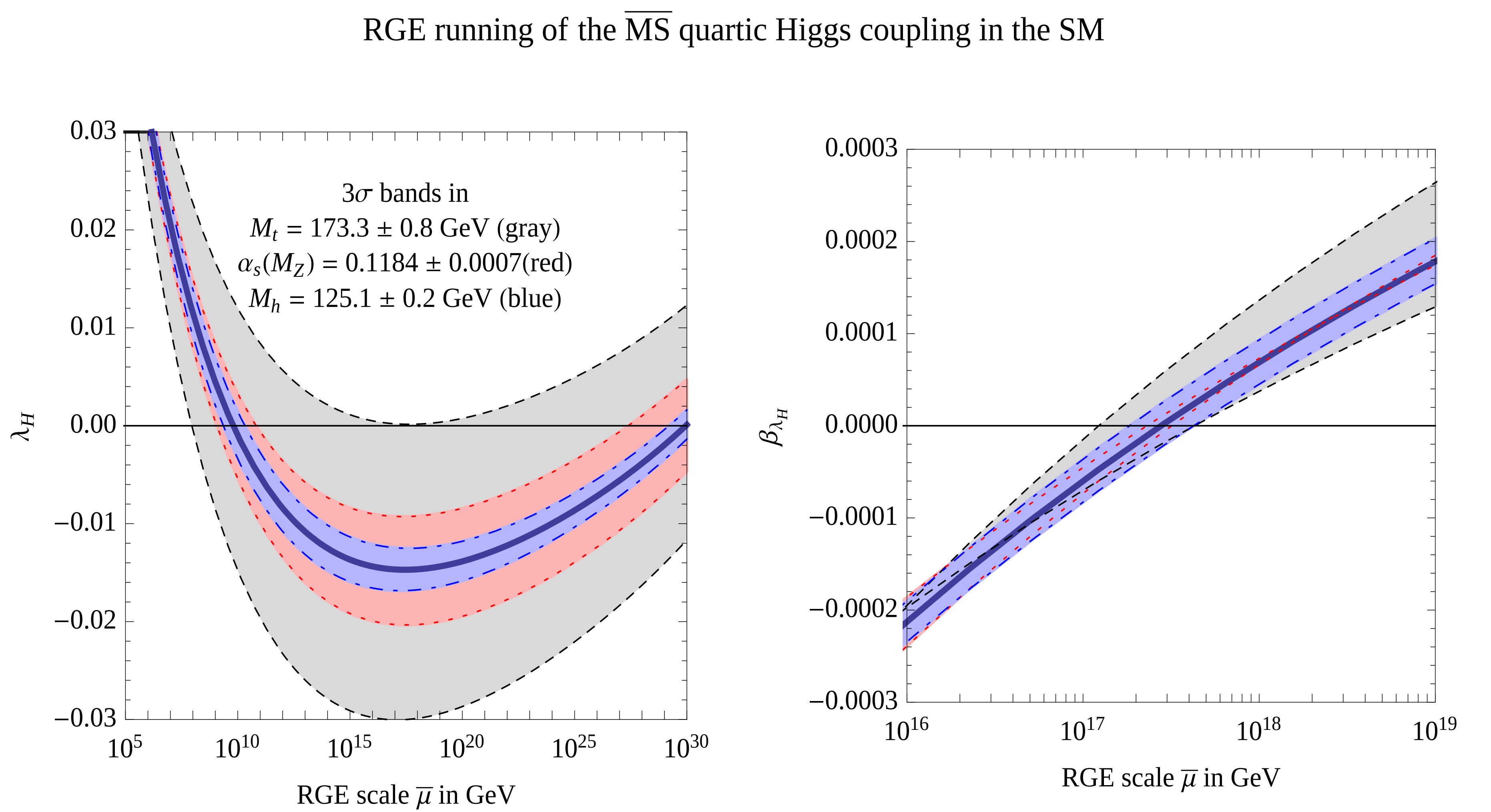
$$\lambda_S(\bar\mu \sim \langle s\rangle) = 0$$

Then the minimum equation simplifies to

$$\beta_{\lambda_S}(\bar\mu \sim \langle s\rangle) = 0$$

Is this fine-tuned running possible?

This is how λ_H runs in the SM



We do not live in the $h \sim 10^{17.5}$ GeV minimum. Another scalar needed: a SM mirror, or something else with gauge and Yukawa interactions.

Generation of the Weak scale

RGE running from the ghost mass $M_{0,2}$ to $M_{\rm Pl}$:

$$(4\pi)^2 \frac{dM_h^2}{d\ln\bar\mu} = -\xi_H \left[5 f_2^4 + f_0^4(1+6\xi_H)\right] \bar M_{\rm Pl}^2 + \cdots$$

The weak scale arises if $f_{0,2} \sim \sqrt{M_h/M_{\rm Pl}} \sim 10^{-8}$ i.e. $M_{0,2} \sim 10^{11}$ GeV

All small parameters such as $f_{0,2}$ and $\lambda_{HS} \sim f_{0,2}^4$ are naturally small

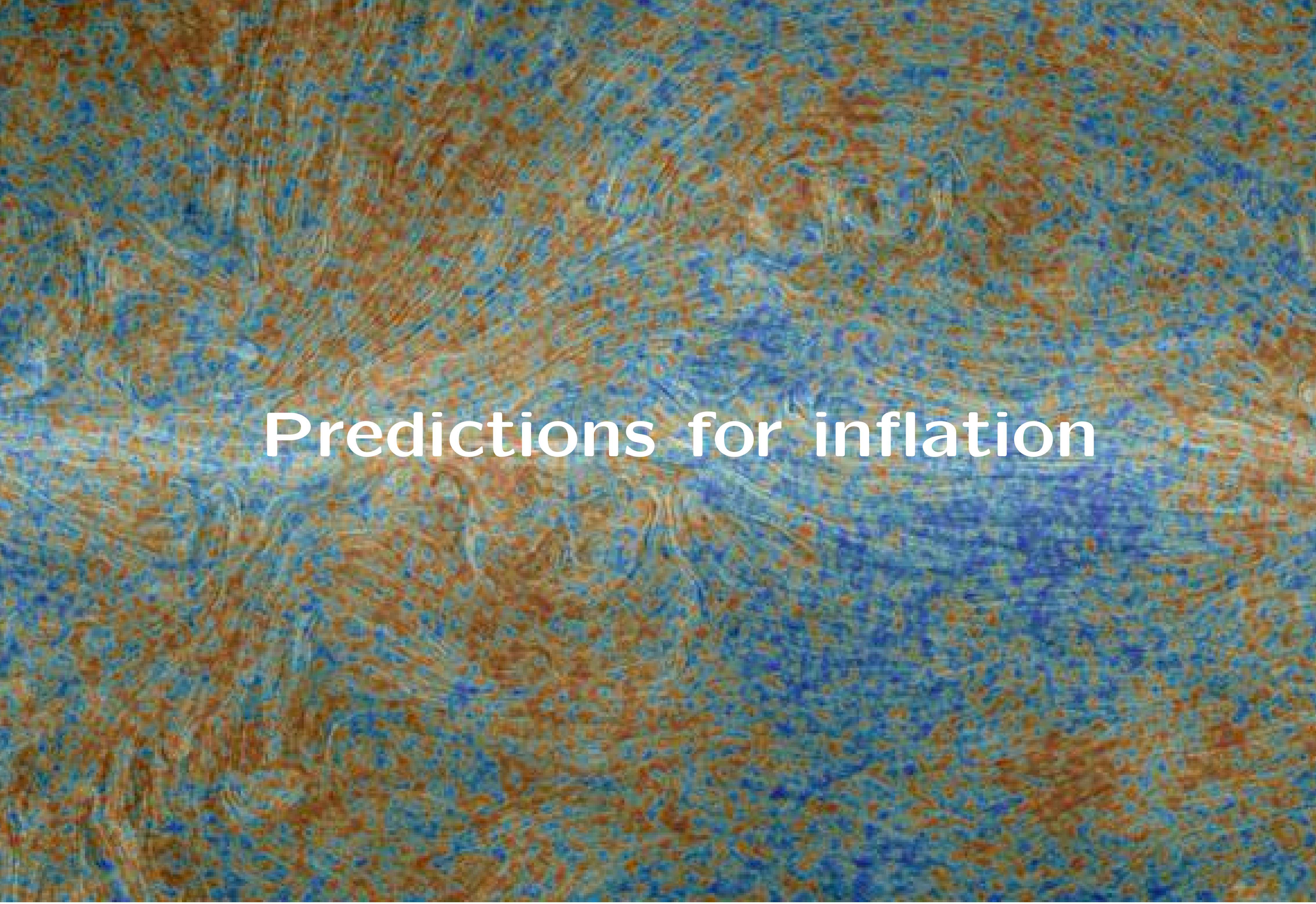
Non-perturbative quantum gravity

Einstein gravity becomes strongly-coupled at $M_{\rm Pl}$. Black holes with mass $M_{\rm BH}\sim M_{\rm Pl}$ can give unnaturally large non-perturbative corrections to M_h:

$$\delta M_h^2 \sim \int M_{\rm BH}^2\, e^{-S}, \qquad S = 4\pi\frac{M_{\rm BH}^2}{M_{\rm Pl}^2}\sim\frac{4\pi}{g_{\rm grav}^2}.$$

In agravity $g_{\rm grav}\to f_{0,2}\lesssim 10^{-8}$, so non-perturbative effects should be negligible. Indeed states with $M_{\rm BH}\lesssim M_{\rm Pl}/f_{0,2}$ get modified by an healtier

$$V_{\rm Newton} = -\frac{GM}{r}\left[1-\frac{4}{3}e^{-M_2 r}+\frac{1}{3}e^{-M_0 r}\right]$$



Predictions for inflation

Inflation = perturbative agravity

Inflation is not a generic phenomenon: one needs to flatten potentials or justify hilltop initial conditions or consider super-Planckian field variations, which are forbidden in string theory where the scalar field space is compact with $M_{\rm Pl}$ size.

Inflation is a generic phenomenon in agravity: V is flat in Planck units if all M and $M_{\rm Pl}$ come from $\langle$scalars$\rangle$. The slow-roll parameters are given by the β-functions, which are small if the theory is perturbative. E.g.

$$\epsilon = \frac{1}{2}\frac{\xi_S}{1+6\xi_S}\left[\frac{\beta_{\lambda_S}}{\lambda_S} - 2\frac{\beta_{\xi_S}}{\xi_S}\right]^2,$$

More technically

Consider a generic inflaton s

$$\mathscr{L} = \sqrt{\det g}\left[-f(s)\frac{R}{2} + \frac{(\partial_\mu s)^2}{2} - V(s) + \cdots\right]$$

Make gravity canonical via a Weyl transformation $g_{\mu\nu} = g^E_{\mu\nu} \times \bar{M}^2_{\rm Pl}/f$:

$$\mathscr{L} = \sqrt{\det g_E}\left[-\frac{\bar{M}^2_{\rm Pl}}{2}R_E + \underbrace{\bar{M}^2_{\rm Pl}\left(\frac{1}{f} + \frac{3f'^2}{2f^2}\right)\frac{(\partial_\mu s)^2}{2}}_{\text{If desired make } s \text{ canonical}} - V_E + \cdots\right]$$

where $V_E = \bar{M}^4_{\rm Pl} V/f^2$ is the Einstein-frame potential. If V and f are generic functions, V_E is generic: ad hoc assumptions were invoked to make V_E flat.

In quantum agravity $f = \xi_S(\bar{\mu} \sim s)s^2$ and $V = \frac{1}{4}\lambda_S(\bar{\mu} \sim s)s^4$

So $V_E = \frac{1}{4}\bar{M}^4_{\rm Pl}\lambda_S(s)/\xi_S(s)^2$ is quasi-flat, even above $M_{\rm Pl}$.

Inflaton candidates in agravity

In agravity all scalars can be inflatons, and there are at least 3 scalars:

s The scalar 'Planckion' that breaks scale invariance generating $M_{\rm Pl}$.
It can be light, being the pseudo-Goldstone boson of scale invariance:

$$M_s \sim g_s^2 M_{\rm Pl}/(4\pi)^2$$

If it is the inflation one has $n_s \approx 0.967$ and $r \approx 0.13$.

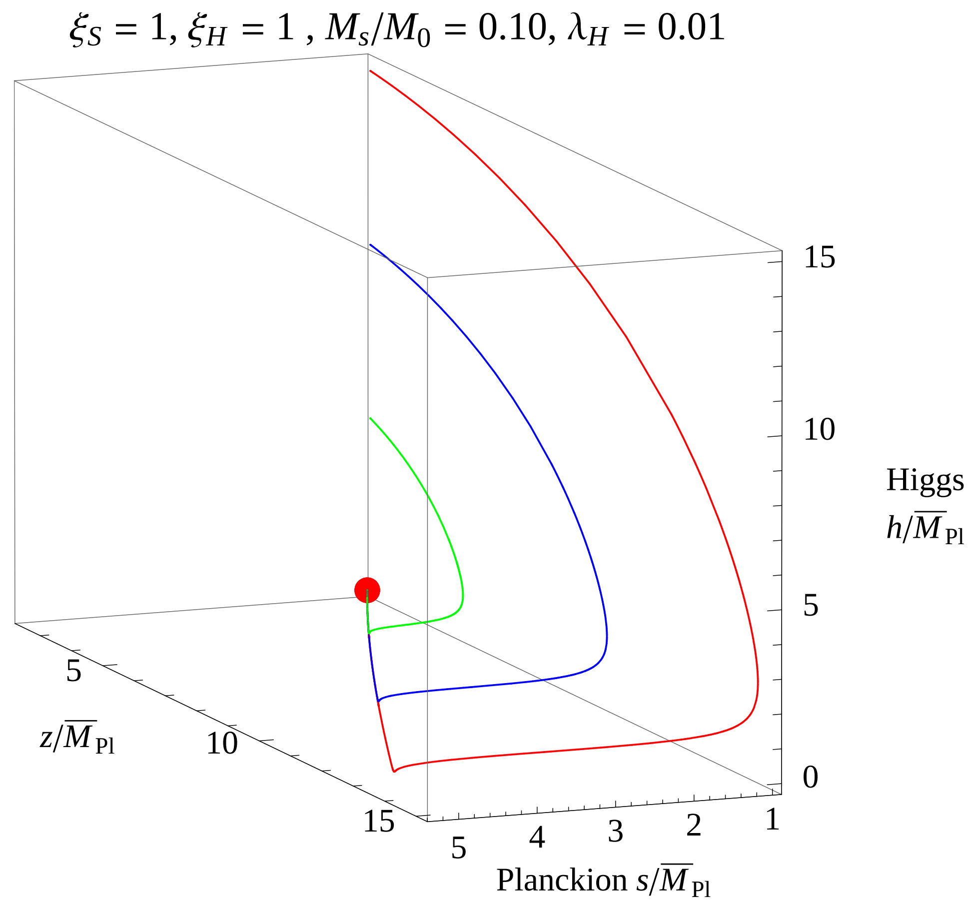
z The scalar component of the graviton, $M_0 \sim f_0 M_{\rm Pl}$.
If it is the inflaton one has Starobinski inflation: $n_s \approx 0.967$ and $r \approx 0.003$.

h The Higgs.
If it is the inflation one has Higgs inflation: $n_s \approx 0.967$ and $r \approx 0.003$?

For the moment we ignore the spin 2 ghost.

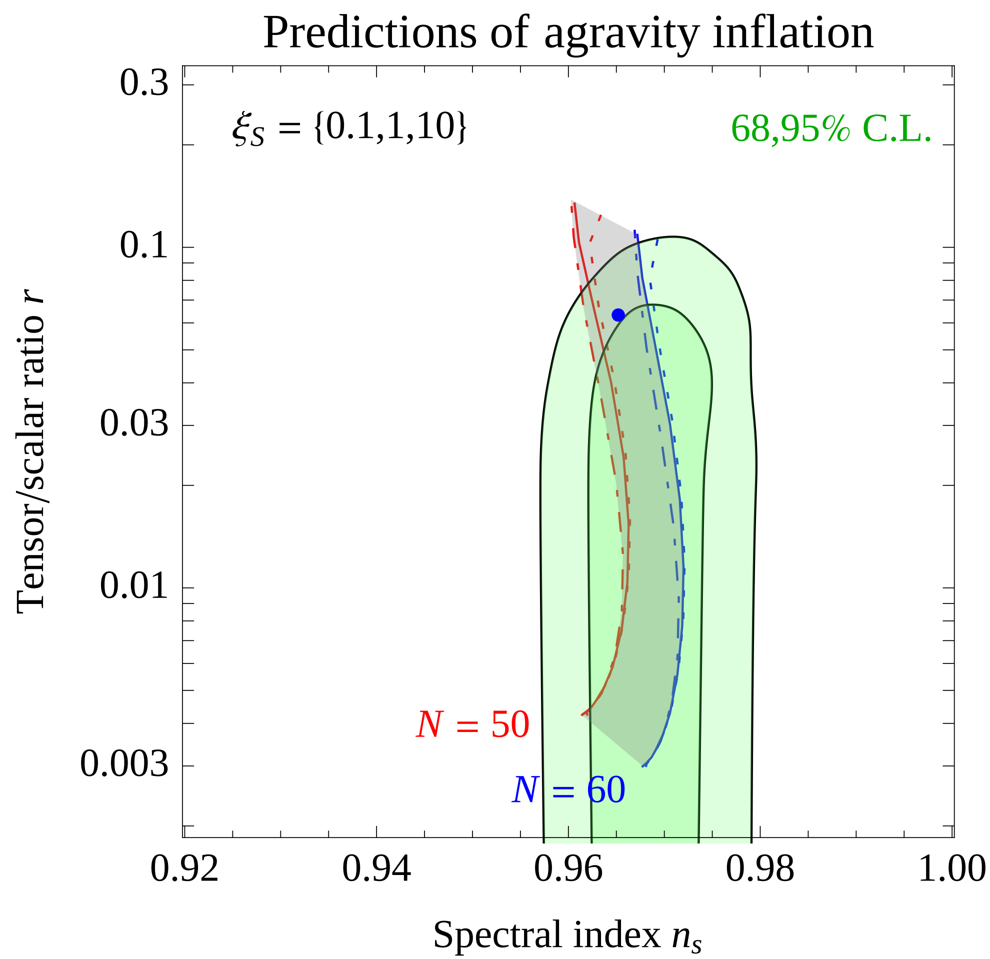
Who is the inflaton?

Predictions might depend on the initial condition. We find that, whatever is the starting point, slow-roll converges towards a **unique attractor solution**, probably because a dimensionless-potential has $V'' \sim \lambda\,\text{field}^2$.



The Higgs is never relevant because of its large λ_H.

Predictions for inflation



$P_R \sim M_h/\bar{M}_{\rm Pl}$

Any super-Planckian theory gives inflation, but don't explain $P_R \sim 10^{-9} \ll 1$.

Agravity relates the smallness of the amplitude of inflationary perturbations P_R to the smallness of $M_h/\bar{M}_{\rm Pl}$, up to couplings and loops and powers of $N \approx 60$.

Consider ainflation in the Starobinski limit:

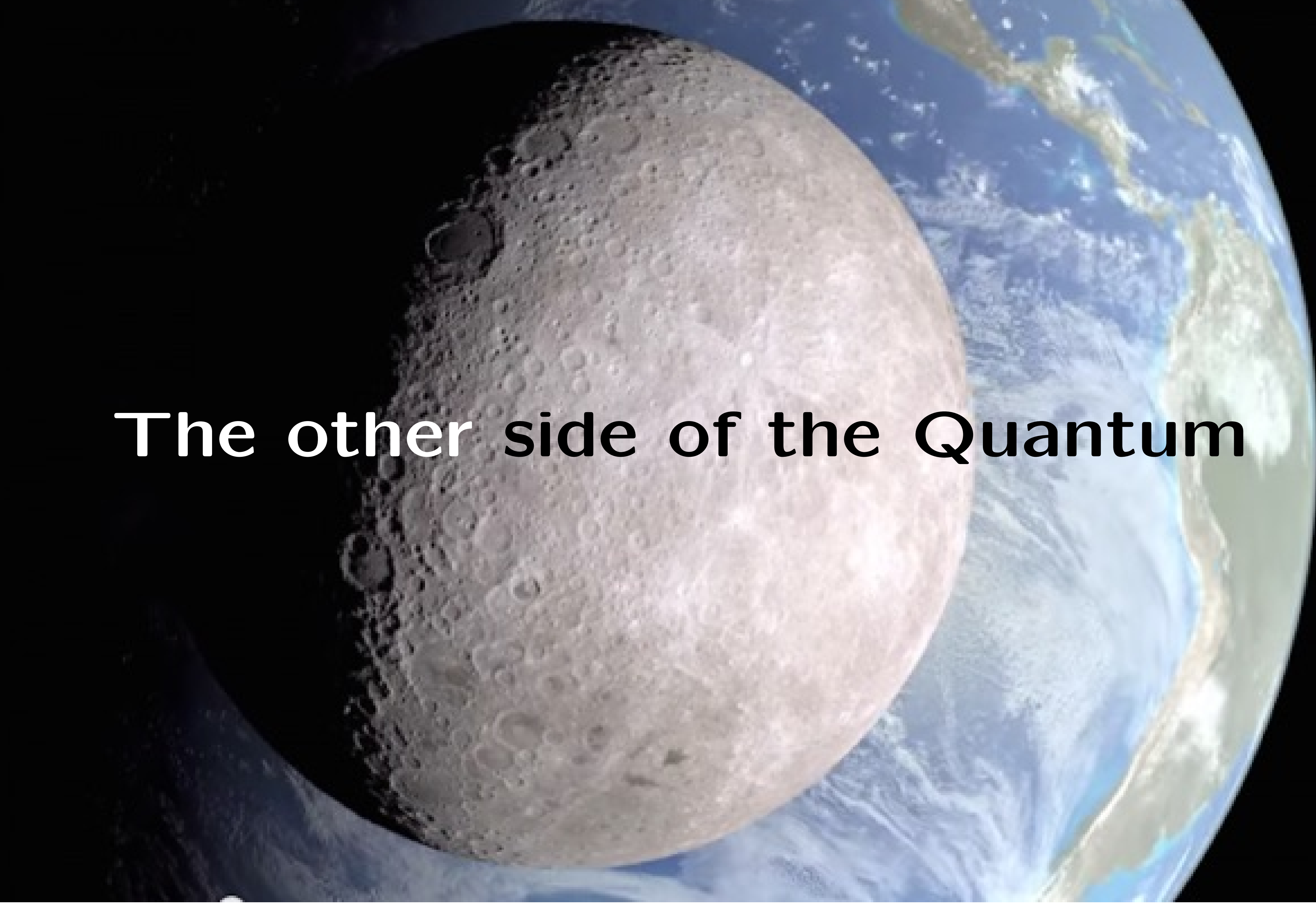
$$P_R = \frac{f_0^2 N^2}{48\pi^2} \qquad \text{i.e.} \qquad f_0 = 1.8\ 10^{-5}.$$

The quantum correction to the Higgs mass is dominated by the RGE:

$$\frac{dM_h^2}{d\ln\bar{\mu}} = -\xi_H(1+6\xi_H) f_0^4 \bar{M}_{\rm Pl}^2 + \cdots$$

So finite naturalness demands $f_0 \lesssim 10^{-5-8}$ (at tree-loop level).

In minimal models, the two values of f_0 are compatible if ξ is close to 0 or $-\frac{1}{6}$.


The other side of the Quantum

Quantisation of 4-derivative systems

Quantisation was first understood for spin 1 and 0 particles with **2 derivatives**.

4 derivative systems have a problem: negative (indefinite) classical H.

Spin 1/2 fields have **1 derivative**. $\mathscr{L} = \bar{\Psi}[i\partial\!\!\!/ - m]\Psi$ classically leads to negative

$$H = \int \frac{d^3p}{(2\pi)^3} E_p[a^\dagger_{p,s} a_{p,s} - b_{p,s} b^\dagger_{p,s}].$$

Quantisation allows positive energy. The two-state solution to $\{b, b^\dagger\} = 1$ shows that one can redefine b into $\tilde{b}^\dagger$ by choosing $|1\rangle$ to have lower energy than $|0\rangle$:

$$b = \begin{array}{c} \\ \langle 0| \\ \langle 1| \end{array}\!\!\begin{array}{c} \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \\ \langle 1| \\ \langle 0| \end{array}\!\!\begin{array}{c} \begin{array}{cc} |1\rangle & |0\rangle \end{array} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{array} = \tilde{b}^\dagger.$$

'Ghosts' are avoided like a plague by serious theorists and explored only by crackpots such as Dirac, Pauli, Heisenberg, Pais and Uhlenbeck, Lee, Wick and Cutkosky, Coleman, Feynman, Boulware and Gross, Hawking and Hertog...

[Salvio, Strumia, 1512.01237]

Ostrogradski no go

Gravity $g_{\mu\nu}(x,t) \quad \approx \quad$ QFT $\phi(x,t) \quad \approx \quad \int_p$ harmonic oscillators in QM... so

Let's focus on a single mode $q(t)$ with 4 time derivatives

$$\mathscr{L} = -\frac{1}{2} q (\frac{d^2}{dt^2} + \omega_1^2)(\frac{d^2}{dt^2} + \omega_2^2) q - V(q)$$

Ostrogradski described the system in canonical form using the auxiliary coordinate $q_2 = \lambda \dot{q}$ with $\lambda = 1$. Keep λ generic:

$$\begin{cases} q_1 = q, & p_1 = \dfrac{\delta S}{\delta \dot{q}_1} = (\omega_1^2 + \omega_2^2)\dot{q} + \dddot{q}, \\ q_2 = \lambda \dot{q}, & p_2 = \dfrac{\delta S}{\delta \dot{q}_2} = -\dfrac{\ddot{q}}{\lambda} \end{cases}$$

The Hamiltonian is unbounded from below

$$H = \sum_{i=1}^{2} p_i \dot{q}_i - \mathscr{L} = \frac{p_1 q_2}{\lambda} - \frac{\lambda^2}{2} p_2^2 - \frac{\omega_1^2 + \omega_2^2}{2\lambda^2} q_2^2 + \frac{\omega_1^2 \omega_2^2}{2} q_1^2 + V(q_1).$$

Classical solution

If $V = 0$, the classical solution has no run-aways

$$q(t) = \frac{a_1 e^{-i\omega_1 t}}{\sqrt{2\omega_1(\omega_1^2 - \omega_2^2)}} + \frac{a_2 e^{-i\omega_2 t}}{\sqrt{2\omega_2(\omega_1^2 - \omega_2^2)}} + \text{h.c.}$$

Indeed the system can be decomposed into two decoupled oscillators with frequencies $\omega_1 > \omega_2$ but opposite-sign energies

$$H = -\frac{1}{2}(\tilde{p}_1^2 \tilde{\lambda}^2 + \omega_1^2 \frac{\tilde{q}_1^2}{\tilde{\lambda}^2}) + \frac{1}{2}(\tilde{p}_2^2 + \omega_2^2 \tilde{q}_2^2)$$

through the canonical transformation

$$\begin{cases} q_1 = \dfrac{\tilde{q}_2 - \tilde{\lambda}\tilde{p}_1/\omega_1}{\sqrt{\omega_1^2 - \omega_2^2}}, & \dfrac{q_2}{\lambda} = \dfrac{\tilde{p}_2 - \omega_1 \tilde{q}_1/\tilde{\lambda}}{\sqrt{\omega_1^2 - \omega_2^2}}, \\ p_1 = \omega_1 \dfrac{\omega_1 \tilde{p}_2 - \omega_2^2 \tilde{q}_1/\tilde{\lambda}}{\sqrt{\omega_1^2 - \omega_2^2}}, & p_2 \lambda = \dfrac{\omega_2^2 \tilde{q}_2 - \omega_1 \tilde{\lambda}\tilde{p}_1}{\sqrt{\omega_1^2 - \omega_2^2}}. \end{cases}$$

Adding interactions between the two oscillators gives run-away solutions.

Quantum will be done perturbatively starting from the free theory.

Keep in mind that making sense of interactions is the real issue.

Quantum

The canonical $[q_i, p_j] = i\delta_{ij}$ implies

$$[a_1, a_1^\dagger] = -1, \qquad [a_2, a_2^\dagger] = 1, \qquad H = -\omega_1 a_1 a_1^\dagger + \omega_2 a_2 a_2^\dagger$$

1. **Positive norm, indefinite energy**. Define $a_1^\dagger|\tilde{0}\rangle = 0$ and $a_2|\tilde{0}\rangle = 0$. Solving the vacuum condition using $p_2 = -i\partial/\partial q_2$ with $q_2 = \lambda\dot{q}$ gives

$$\psi_{\tilde{0}}(q_1, q_2) = \exp\left(-\frac{q_1^2\omega_1\omega_2 + q_2^2/\lambda^2}{2}(\omega_1 - \omega_2) + iq_1\frac{q_2}{\lambda}\omega_1\omega_2\right)$$

   which is normalizable for real λ. But excited states have negative energy.

2. **Indefinite norm, positive energy**. Define $a_1|0\rangle = 0$ and $a_2|0\rangle = 0$. The ground-state wave function is non-normalizable for real λ:

$$\psi_0(q_1, q_2) \propto \exp\left(\frac{-q_1^2\omega_1\omega_2 + q_2^2/\lambda^2}{2}(\omega_1 + \omega_2) - iq_1\frac{q_2}{\lambda}\omega_1\omega_2\right).$$

   And imaginary λ gives an apparently bad $H = p_1 q_2/\lambda + \cdots$. Dead?

Negative-norm quantum mechanics

Written as matrix components, it looks unusual with extra confusing $\pm$. Better to consider a generic Hilbert-like space metric $\langle {}_n | {}_m \rangle = \eta_{nm}$, and use an abstract operator formulation that will look identical to usual QM. Define

- Usual bra-ket; inverse metric $\eta^{nm} \equiv (\eta^{-1})_{nm}$;
- Contro-variant basis $|^n\rangle = \eta^{nm}|_m\rangle$. So $\langle^n|^m\rangle = \eta^{nm}$ and $\langle^n|_m\rangle = \delta^n_m = \langle_n|^m\rangle$.
- Components of a state $|\psi\rangle$: $\psi_n \equiv \langle_n|\psi\rangle$, $\psi^n \equiv \langle^n|\psi\rangle$. So $|\psi\rangle = \psi^n|_n\rangle = \psi_n|^n\rangle$.

- Operator $A = A^{nm}|_n\rangle\langle_m| = A_{nm}|^n\rangle\langle^m| = A_n{}^m|^n\rangle\langle_m| = A^n{}_m|_n\rangle\langle^m|$:

$$A_{nm} \equiv \langle_n|A|_m\rangle, \qquad A^{nm} \equiv \langle^n|A|^m\rangle, \qquad A_n{}^m \equiv \langle_n|A|^m\rangle, \qquad A^n{}_m \equiv \langle^n|A|_m\rangle.$$

- Unity operator $1 = \eta^{nm}|_n\rangle\langle_m| = \eta_{nm}|^n\rangle\langle^m| = |_n\rangle\langle^n| = |^n\rangle\langle_n|$.
- Operator multiplication: $(AB)_{nm} = A_{nn'}\eta^{n'm'}B_{m'm}$, $(AB)_n{}^m = A_n{}^k B_k{}^m$.

- The adjoint $A^\dagger$ such that $|\psi'\rangle = A|\psi\rangle$ implies $\langle\psi'| = \langle\psi|A^\dagger$ is represented by

$$(A^\dagger)_{nm} = A^*_{mn} \qquad (A^\dagger)^{nm} = A^{mn*} \qquad (A^\dagger)_n{}^m = (A^m{}_n)^*$$

Mixed components are Hermitian up to an iso-spectral transformation $A_n{}^m = (\eta A^{*T}\eta^{-1})_n{}^m$. A self-adjoint $A^\dagger = A$ has real expectation values $\langle\psi|A|\psi\rangle/\langle\psi|\psi\rangle$.

- Eigenvector equation $H|\psi\rangle = E_\psi|\psi\rangle$:

$$H_n{}^m\psi_m = H_{nm'}\eta^{m'm}\psi_m = E_\psi\psi_n \qquad \text{or} \qquad H^n{}_m\psi^m = \eta^{nn'}H_{n'm}\psi^m = E_\psi\psi^n.$$

- The identity $\langle E_n|H|E_m\rangle = \langle E_n|E_m\rangle E_m = E_n^*\langle E_n|E_m\rangle$ tells that a self-adjoint H can have three different kinds of eigenstates:

+) orthogonal eigenstates $\langle E_n|E_m\rangle = 0$ with real E_n and $\langle E_n|E_n\rangle = +1$;

0) orthogonal eigenstates $\langle E_n|E_m\rangle = 0$ with real E_n and $\langle E_n|E_n\rangle = -1$;

−) pairs of complex conjugated eigenvalues, $E_n = E_m^*$ with $\langle E_n|E_m\rangle \neq 0$ and zero norm, $\langle E_n|E_n\rangle = 0$. Looks bad, like tachions in positive-norm.

Time evolution

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \qquad \Rightarrow \qquad |\psi(t)\rangle = U(t)|\psi(0)\rangle \quad \text{with} \quad U(t) = \mathrm{T}e^{-iHt}$$

Time evolution is unitary, $U^\dagger U = 1$, if H is self-adjoint, $H^\dagger = H$

Usual proof: $i\frac{\partial}{\partial t}\langle\psi'(t)|\psi(t)\rangle = \langle\psi'|H - H^\dagger|\psi\rangle = 0$

Unusual in components:

$$i\frac{\partial}{\partial t}\psi_n = H_{nm}\eta^{mm'}\psi_{m'} \qquad \text{or} \qquad i\frac{\partial}{\partial t}\psi^n = \eta^{nn'}H_{nm}\psi^m.$$

${U_n}^m$ is the usual naive exponentiation of ${H_n}^m$, which is not a hermitian matrix.
H_{nm} is hermitian, but U_{nm} is not its naive exponentiation. Rather

$$U_{nm} = \left[\eta + \eta(-iHt)\eta + \frac{1}{2}\eta(-iHt)\eta(-iHt)\eta + \cdots\right]_{nm}$$

Unitarity in components: usual $U^{k*}_{\ n}{U_k}^m = \delta^n_m$ and unusual $U^*_{n'n}\eta^{n'm'}U_{m'm} = \eta_{nm}$.

The indefinite-norm two-state system

Generic H (up to trivialities):

$$H = \frac{1}{2} \begin{array}{c} \\ \langle + | \\ \langle - | \end{array} \begin{array}{cc} |+\rangle & |-\rangle \\ \multicolumn{2}{c}{\begin{pmatrix} E_R & -iE_I \\ iE_I & E_R \end{pmatrix}} \end{array} = \frac{1}{2} \begin{array}{c} \\ \langle + | \\ \langle - | \end{array} \begin{array}{cc} |^{+}\rangle & |^{-}\rangle \\ \multicolumn{2}{c}{\begin{pmatrix} E_R & +iE_I \\ iE_I & -E_R \end{pmatrix}} \end{array}$$

Eigenvalues $E_\pm = \pm\sqrt{E_R^2 - E_I^2}/2$, eigenvectors:

$$|E_+\rangle = \sqrt{\frac{\gamma+1}{2}}|+\rangle - i\sqrt{\frac{\gamma-1}{2}}|-\rangle, \qquad |E_-\rangle = i\sqrt{\frac{\gamma-1}{2}}|+\rangle + \sqrt{\frac{\gamma+1}{2}}|-\rangle$$

where $\gamma = 1/\sqrt{1 - E_I^2/E_R^2}$ is the mixing ~~angle~~ boost. 3 possible cases:

+) $E_I < E_R$, $|\langle \pm | U | \pm \rangle|^2$ oscillates from 1 **up to** $\gamma^2 \geq 1$.

0) $E_I = E_R$, critical case, zero norm, non-degenerate H

$$U = \begin{array}{c} \\ \langle + | \\ \langle - | \end{array} \begin{array}{cc} |^{+}\rangle & |^{-}\rangle \\ \multicolumn{2}{c}{\begin{pmatrix} 1 - iE_R t/2 & E_R t/2 \\ E_R t/2 & 1 + iE_R t/2 \end{pmatrix}} \end{array}$$

−) $E_I > E_R$, pair of complex eigenvalues $E_\pm$ with runaway in U $|\psi(t)\rangle = \psi^{E_+} e^{-iE_+ t} |E_+\rangle + \psi^{E_-} e^{-iE_- t} |E_-\rangle$. Constant $\langle \psi(t) | H | \psi(t) \rangle$ and $\langle \psi(t) | \psi(t) \rangle$.

Perturbation theory

$H = H_0 + V(t)$ can be solved perturbatively with the usual Interaction picture

$$U_I(t_i, t_f) = \mathsf{T}\, e^{-i \int_{t_i}^{t_f} dt\, V_I(t)} = 1 - i \int_{t_i}^{t_f} dt'\, V_I(t') + \cdots$$

The $\int dt\, e^{-i(E_i - E_f)t}$ means that the energy conserved by quantum evolution (up to $\Delta t\, \Delta E \geq \hbar$) are the eigenvalues of H. $\langle \psi | H | \psi \rangle$ can be negative without giving problems, just imagine going in the basis where H is diagonal.

The negative-norm harmonic oscillator

Consider arbitrary signs s_H and s_C:

$$H = s_H \frac{a^\dagger a + a a^\dagger}{2}, \qquad [a, a^\dagger] = s_C.$$

Define $a|0\rangle = 0$ and $|n\rangle = a^\dagger |n-1\rangle / \sqrt{n}$. Implied metric: $\eta_{nm} \equiv \langle m|n\rangle = s_C^n \delta_{nm}$.

Run-aways are avoided if the Hamiltonian eigenvalues

$$H|n\rangle = E_n |n\rangle \qquad E_n = (n + \frac{1}{2}) s_C s_H$$

are positive. There are two solutions:

- The usual positive H and positive norm, $s_C = s_H = +1$.
- Negative H and indefinite norm, $s_C = s_H = -1$.

Some authors look at matrices and improperly tell that q, p are 'anti-Hermitian'.

$${a_n}^m = s \times \text{ (the hermitian conjugate of } {(a^\dagger)_n}^m).$$

So $q = (a + a^\dagger)/\sqrt{2}$ and $p = i(a^\dagger - a)/\sqrt{2}$ are represented by anti-Hermitian matrices ${q_n}^m$ and ${p_n}^m$. **But q and p are self-adjoint**.

A small interaction leaves the negative norm system good (no complex E_n).

Negative-norm coordinate representation

$|n\rangle$ has norm and parity equal to $(-1)^n$. So

$$\langle\psi'|\psi\rangle = \int dx\,[\psi'^*_{\text{even}}(x)\psi_{\text{even}}(x) - \psi'^*_{\text{odd}}(x)\psi_{\text{odd}}(x)] = \int dx\,\psi'^*(x)\psi(-x)$$

i.e. $1 = \int dx|x\rangle\langle x|$ with $|x\rangle = |-x\rangle$ i.e. $\langle x'|x\rangle = \delta(x + x')$.

A strange beast already known to Dirac-Pauli emerges:

Indefinite-norm coordinate representation

$$q|x\rangle = ix|x\rangle, \qquad p|x\rangle = +\frac{d}{dx}|x\rangle.$$

q and p are self-adjoint. Explicit check:

$$\langle x'|q^\dagger|x\rangle = \langle x|q|x'\rangle^* = [ix'\delta(x+x')]^* = ix\delta(x+x') = \langle x'|q|x\rangle$$

Solve $\langle x|a|0\rangle = 0$ with $a = (q+ip)/\sqrt{2}$: $\psi_0 \propto e^{-x^2/2}$ is normalisable.

norm	$\langle x\|q\|\psi\rangle$	T-parity	$\langle x\|p\|\psi\rangle$	T-parity	harmonic oscillator with $E > 0$
positive	$x\psi(x)$	even	$-i\,d\psi/dx$	odd	$\psi_0(q) \propto e^{-q^2/2}$ and $H = +\frac{1}{2}(q^2+p^2)$
indefinite	$-ix\psi(x)$	odd	$d\psi/dx$	even	$\psi_0(q) \propto e^{-q^2/2}$ and $H = -\frac{1}{2}(q^2+p^2)$

4 derivatives want Dirac-Pauli

Classically: $q_1 = q$ and auxiliary $q_2 = \lambda \dot{q} = \dot{q}$ as natural choice.

At quantum level $|q_1, q_2\rangle$. T-even q implies T-odd $\dot{q}$. Equivalent quantisations:

1) Define $q_2 = i\dot{q}$ to make q_2 T-even and use the usual representation.
(not so strange: analogous to $p = i\boldsymbol{\nabla}$ rather than $p = \boldsymbol{\nabla}$)

2) Use the naturally T-odd indefinite-norm coordinate representation for $\dot{q}$.
(Principled)

The issue is interactions.

In approach 1) $H(q_1, q_2, p_1, p_2)$ contains i. Unitarity???

In approach 2) all q, p are self-adjoint, so any real H gives a 'unitary' theory.

Path integral

Generalisation to indefinite metric is immediate

$$\langle^{q_f,t_f}|_{q_i,t_i}\rangle = \int Dq\,Dp\; e^{i\int dt[p\dot{q}-H_{\text{cl}}]} \qquad H_{\text{cl}} \equiv \frac{\langle p|H|q\rangle}{\langle p|q\rangle}$$

Apply to 4-derivative theories. This means

1) A propagator $\langle q_f, -\dot{q}_f, t_f | q_i, \dot{q}_i, t_i\rangle$ with an unusual $-$ in its external state.

This is equivalent to using the indefinite norm: indeed for $t_f \to t_i$ one has $\langle^{q_f,\dot{q}_f}|_{q_i,\dot{q}_i}\rangle = \delta(q_f - q_i)\delta(\dot{q}_f - \dot{q}_i)$.

Furthermore, the T-odd nature of $\dot{q}$ is hardwired in the path integral

2) A classical Hamiltonian going into the complex plane

$$H_{\text{cl}} = \frac{\langle p_1,p_2|H|q_1,q_2\rangle}{\langle p_1,p_2|q_1,q_2\rangle} = ip_1q_2 + \frac{p_2^2}{2} + \frac{\omega_1^2+\omega_2^2}{2}q_2^2 + \frac{\omega_1^2\omega_2^2}{2}q_1^2.$$

Lagrangian path integral

Formally, one can do the (divergent) Dp_1, getting a (vanishing) $\delta(q_2 + i\dot{q}_1)$ and recovering the Lagrangian path integral (with run-away classical solutions).

Or, one can first do the Euclidean continuation, $it = t_E$ i.e. $\dot{q} = iq'$, getting

$$\langle^{q_{1f},q_{2f},t_{Ef}}|_{q_{1i},q_{2i},t_{Ei}}\rangle \propto \int Dq_1 Dq_2 Dp_1 Dp_2 \ \exp\left[\int dt_E(ip_1 q_1' + ip_2 q_2' - H_{\text{cl}})\right].$$

Next the Dp_1 intergal is ok, giving $\delta(q_2 - q_1')$ and the well defined Lagrangian Euclidean path-integral:

$$\langle^{q_f,q_f',t_{Ef}}|_{q_i,q_i',t_{Ei}}\rangle \propto \int Dq \ \exp\left[-\int dt_E \ \mathscr{L}_E(q)\right].$$

Even the free theory has classical run-aways, as in any Euclidean theory:

$$q_{\text{cl}}(t_E) = a_1 e^{-\omega_1 t_E} + a_2 e^{-\omega_2 t_E} + b_1 e^{\omega_1 t_E} + b_2 e^{\omega_2 t_E}.$$

Its action gives the normalisable ground state wave function

$$\langle q, q', t_E = 0 | 0, 0, t_E = -\infty\rangle \propto \exp\left[-\frac{q^2\omega_1\omega_2 + q'^2}{2}(\omega_1 + \omega_2) + qq'\omega_1\omega_2\right].$$

which reproduces the Minkowski operator wave function $\psi_0(q_1 = q, q_2 = q')$. Hawking-Hertog get a divergent ψ because they instead continue to $\dot{q} = iq'$.

Interactions

Interactions are ok: as usual one just needs a real $H(q,p)$ without tachyons. For example harmonic oscillators + small interactions.

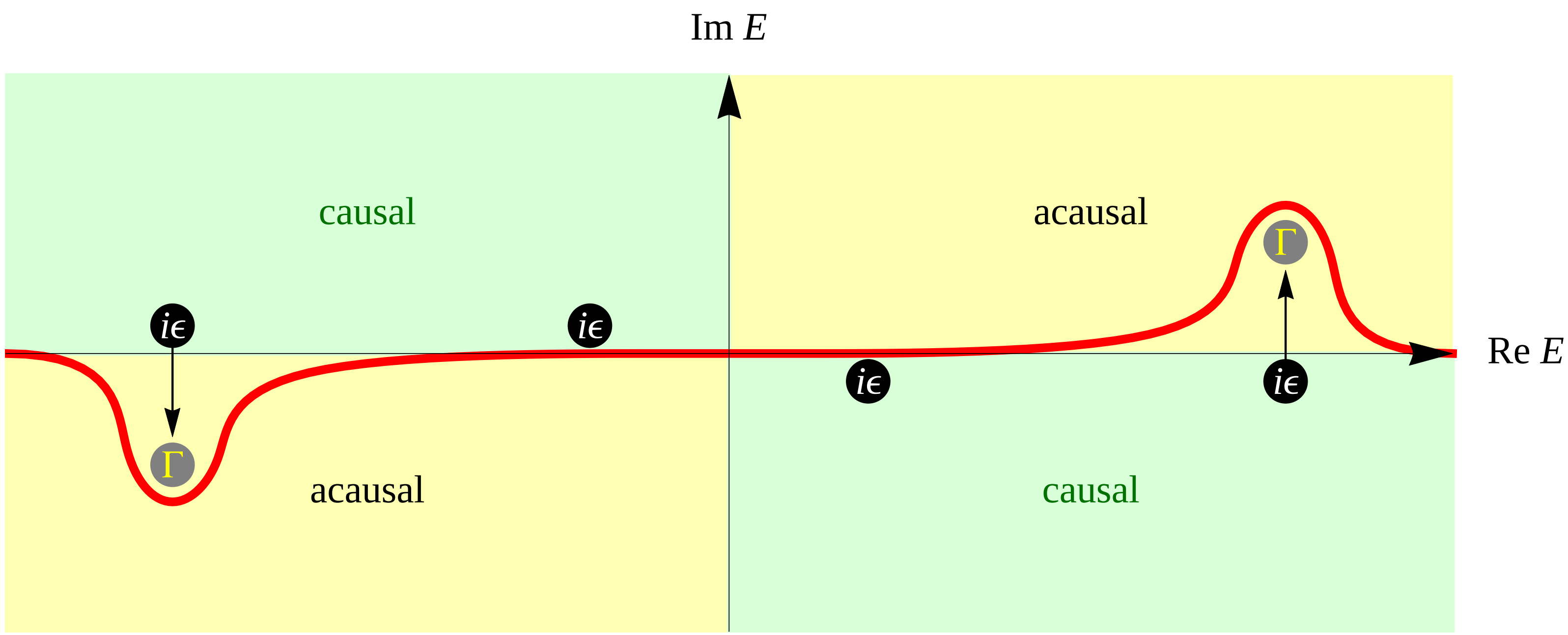
States with $\langle\psi|H|\psi\rangle < 0$? ok. What enters in conservation is H eigenvalues.

Heisenberg formalism? $\dot{A} = -i[A,H]$ looks classical (sick), but operator $\neq$ number. Solved by $A(t) = U^\dagger(t)A(0)U(t)$ even for $A = \dot{q}$, equivalent to Schrödinger.

Problems: QFT?

Propagator? The naive $\langle 0|Tq(t)q(t')|0\rangle = \int \frac{dE}{2\pi} \frac{-i\, e^{-iE(t-t')}}{(E^2-\omega_1^2+i\epsilon)(E^2-\omega_2^2+i\epsilon)}$.

QFT could be problematical. Continuum implies that a massive ghost X is degenerate with particles: decay $X \to \gamma\gamma$, so tachyons could develop. $\Pi = -(p^2-m_1^2)(p^2-m_2^2)$ gets positive imaginary part. Acausal decay???



Problems: probability interpretation?

Meaning of quantum states that entangle positive-norm with negative norm? QM is deterministic + the probabilitistic (!??) Copenhagen interpretation.

Lee-Wick: all asymptotic stable states can have positive norm.

Any self-adjoint H gives unitary evolution with respect to many different norms: each energy eigenstate picks a phase. Redefine to positive norm defining

ghost parity $\equiv \mathcal{G} \equiv \eta_{mn}$ in the special basis of energy eigenstates

and $|^{\psi}\rangle = \mathcal{G}^{-1}|_{\psi}\rangle$ and

$$P_n = \langle^{\psi}|\Pi_n|_{\psi}\rangle \qquad \text{where} \qquad \Pi_n = |^{n}\rangle\langle_{n}|.$$

Upward oscillations become downward; real $\langle_{\psi}|A|_{\psi}\rangle$ is not probabilistic; complex $\langle^{\psi}|A|_{\psi}\rangle$ has a probabilistic interpretation.

Ghost summary

A pair of canonical coordinates (q, p) admits two coordinate representations:

	$\widehat{q}$	T	$\widehat{p}$	T	norm	harmonic oscillator
visibile face	q	even	$-i\partial/\partial q$	odd	$+$	$\psi_0 \propto e^{-q^2/2}$ if $H = +a^\dagger a$
hidden face	iq	odd	$-\partial/\partial q$	even	$\pm$	$\psi_0 \propto e^{-q^2/2}$ if $H = -a^\dagger a$

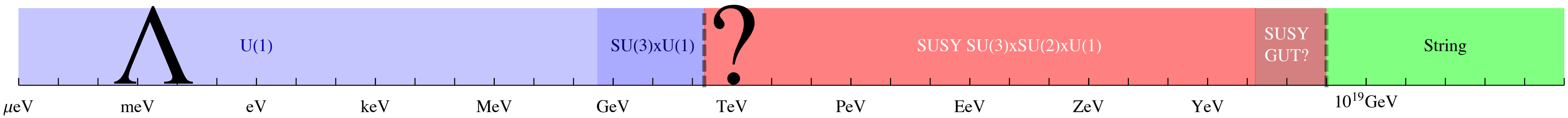
In both cases, q, p, H are self-adjoint and the eigenvalues of H are positive.

4-derivative $q(t)$ can be rewritten as two 2-derivative canonical $q_1 = q$ and $q_2 = \dot{q}$, which is naturally T odd, and must follow the negative norm quantisation.

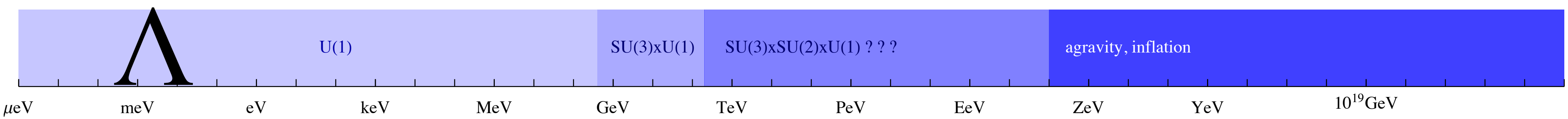
With negative norm, self-adjoint means hermitian up to a self-similar transformation, and gives 'unitary' evolution. Interactions ok. QFT? Probability??

Conclusions

The standard view of mass scales in nature is in trouble with M_h and Λ:



New collider needed to fully clarify. Possible alternative for M_h:



Quantisation of $\partial^4 \Rightarrow$ physical naturalness + quantum gravity + inflation.

Remaining problems: give an interpretation to ‘ghosts’.