How the huge energy of quantum vacuum gravitates to drive the slow accelerating expansion of the Universe

VIA Lecture
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Uncertainty Principle of Quantum Mechanics

Equivalence Principle of General Relativity

Huge density of vacuum energy

Every form of energy gravitates

How vacuum gravitates?

Uncertainty Principle of Quantum Mechanics

↓

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Before the discovery of the **accelerating expansion of the universe:**

Old cosmological constant problem: why effective cosmological constant is not large?

After the discovery of the **accelerating expansion of the universe:**

New cosmological constant problem: why effective cosmological constant is not zero but a specific small value?

Steven Weinberg (1989): **a veritable crisis**

Edward Kolb and Michael Turner (1993): **an unexplained puzzle**

D. Dolgov (1997): **the most striking problem in contemporary physics**

Edward Witten (2001): **one of the major obstacles to further progress in fundamental physics**

Leonard Susskind (2015): **the mother of all physical problems, the worst predictions ever!**

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Maybe we need to be more Careful about what QM and GR really tell us about "how vacuum gravitates"?

Key steps in formulating the cosmological constant problem

In some literature, using directly the Einstein equations

$$G_{\mu\nu} + \lambda_b g_{\mu\nu} = 8\pi G T_{\mu\nu}^{\mathrm{vac}}$$

And assuming vacuum equation of state

$$T_{\mu\nu}^{\mathrm{vac}}(t, \mathbf{x}) = -\rho^{\mathrm{vac}} g_{\mu\nu}(t, \mathbf{x})$$

In other literature, assuming Semiclassical Einstein equations

$$G_{\mu\nu} + \lambda_b g_{\mu\nu} = 8\pi G \left\langle T_{\mu\nu}^{\mathrm{vac}} \right\rangle$$

And assuming vacuum equation of state for expectation values

$$\langle T_{\mu\nu}(t, \mathbf{x}) \rangle = -\rho^{\mathrm{vac}} g_{\mu\nu}(t, \mathbf{x})$$

Identify vacuum with the cosmological constant by

$$\lambda_{\mathrm{eff}} = \lambda_b + 8\pi G \rho^{\mathrm{vac}}$$

or

$$\rho_{\mathrm{eff}}^{\mathrm{vac}} = \rho^{\mathrm{vac}} + \frac{\lambda_b}{8\pi G}$$

ρ^{vac} has to be a constant as required by the conservation of stress energy tensor

$$\nabla^{\mu} T_{\mu\nu}^{\mathrm{vac}} = 0$$

So in the usual formulation of the Cosmological constant problem, the quantum vacuum is considered to be homogeneous and isotropic

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

Einstein equations

$$3H^2 = \lambda_{\mathrm{eff}} = 8\pi G \rho_{\mathrm{eff}}^{\mathrm{vac}},$$

$$\ddot{a} = \frac{\lambda_{\mathrm{eff}}}{3} a_{\mathrm{vac}} = \frac{8\pi G \rho_{\mathrm{eff}}^{\mathrm{vac}}}{3} a \geq 0.$$

Solution

$$a(t) = a(0) e^{Ht}$$

where $$H = \frac{\dot{a}}{a} = \pm\sqrt{\frac{8\pi G \rho_{eff}^{vac}}{3}} \propto \sqrt{G}\Lambda^2 \to \infty, \quad \text{as taking} \quad \Lambda \to \infty$$

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However, the density of vacuum energy can never be a constant. It comes from vacuum fluctuations, but the magnitude of vacuum fluctuations itself also fluctuates.

Constantly fluctuating and extremely inhomogeneous quantum vacuum

Consider a massless scalar field in vacuum state $|0\rangle$

$$\phi(t,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a_{\mathbf{k}} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} + a_{\mathbf{k}}^{\dagger} e^{+i(\omega t - \mathbf{k}\cdot\mathbf{x})} \right) \qquad a_{\mathbf{k}}|0\rangle = 0, \quad \text{for all } \mathbf{k}$$

Vacuum $|0\rangle$ is an eigenstate of the total Hamiltonian, but Not an eigenstate of the enegy density operator.

$$\mathrm{H} = \int d^3x\, T_{00} = \frac{1}{2} \int d^3k\, \omega \left(a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} + a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \right)$$

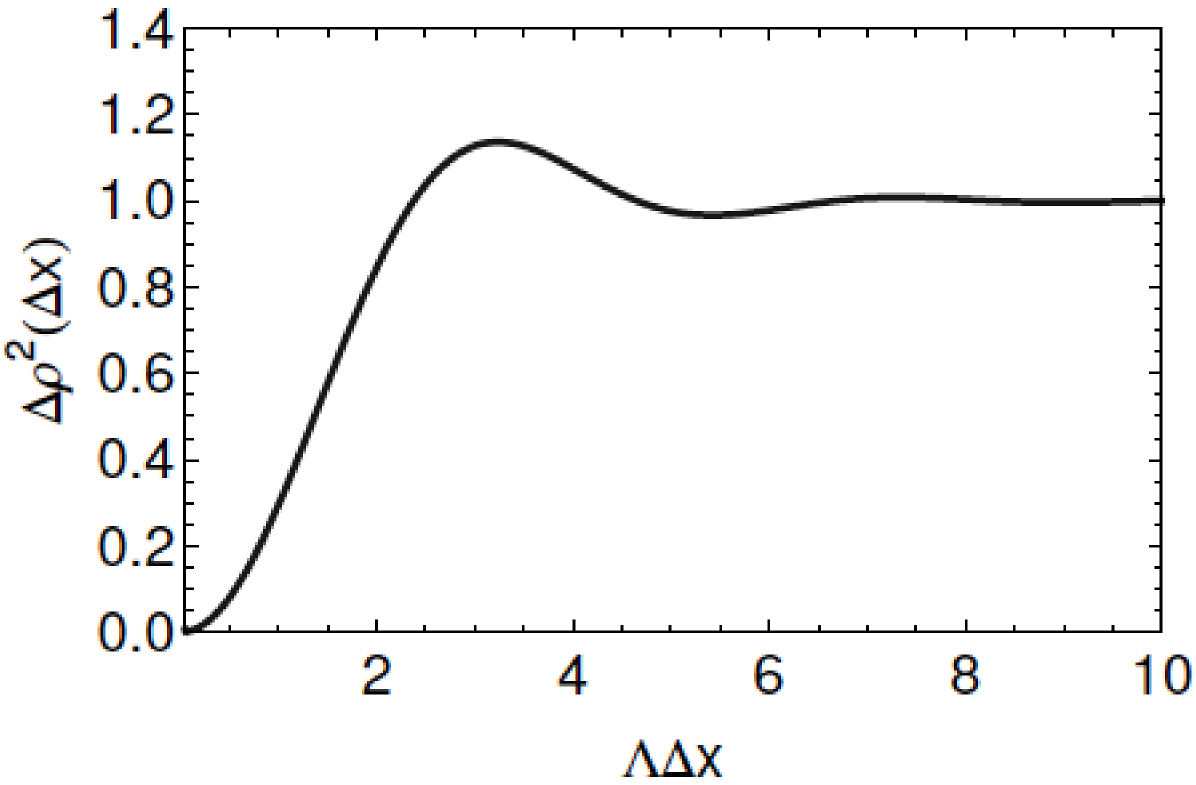
$$T_{00}(t,\mathbf{x}) = \frac{1}{2} \int \frac{d^3k d^3k'}{(2\pi)^3} \frac{1}{2} \left(\sqrt{|\mathbf{k}||\mathbf{k}'|} + \frac{\mathbf{k}\cdot\mathbf{k}'}{\sqrt{|\mathbf{k}||\mathbf{k}'|}} \right) \left(a_{\mathbf{k}} a_{\mathbf{k}'}^{\dagger} e^{-i[(|\mathbf{k}|-|\mathbf{k}'|)t - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}]} \right.$$

$$\left. + a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}'} e^{+i[(|\mathbf{k}|-|\mathbf{k}'|)t - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}]} - a_{\mathbf{k}} a_{\mathbf{k}'} e^{-i[(|\mathbf{k}|+|\mathbf{k}'|)t - (\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}]} - a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}'}^{\dagger} e^{+i[(|\mathbf{k}|+|\mathbf{k}'|)t - (\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}]} \right)$$

Huge fluctuations $\left\langle \left(T_{00} - \langle T_{00} \rangle \right)^2 \right\rangle = \frac{2}{3} \langle T_{00} \rangle^2$ $\qquad \langle T_{00} \rangle = \frac{\Lambda^4}{16\pi^2} \to +\infty$

Λ is the effective QFT's high energy cutoff

Extreme inhomogeneity



$$\Delta\rho^2(\Delta x) = \frac{\left\langle \left\{ \left(T_{00}(t,\mathrm{x}) - T_{00}(t,\mathrm{x}')\right)^2 \right\} \right\rangle}{\frac{4}{3}\langle T_{00}(t,\mathrm{x})\rangle^2} \qquad \Delta x = |\mathrm{x} - \mathrm{x}'|$$

Difference in energy density is on the same order of $\langle T_{00} \rangle$ itself for spatial separations larger than $\Delta x \sim 1/\Lambda, \quad \Lambda \to +\infty$

Since the vacuum is extremely inhomogeneous, the usual FLRW metric is not valid to describe its gravitational behaviour. And the de Sitter solution

$$a(t) = a(0)e^{Ht} \qquad \text{where} \quad H = \frac{\dot{a}}{a} = \pm\sqrt{\frac{8\pi G \rho_{\rm eff}^{\rm vac}}{3}}$$

may no longer be valid.

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may no longer be valid.

We need a new method to investigate the gravity of vacuum.

Also, the extreme inhomogeneity of vacuum means that the gravity of vacuum can **Not** be treated **Perturbatively**.

A simple model

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$\Downarrow$$

$$ds^2 = -dt^2 + a^2(t, \mathrm{x})(dx^2 + dy^2 + dz^2)$$

Differences made by the inhomogeneous vacuum

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G T_{00},$$
$$G_{ii} = -2a\ddot{a} - \dot{a}^2 = 8\pi G T_{ii},$$
$$G_{0i} = 0 = 8\pi G T_{0i},$$
$$G_{ij} = 0 = 8\pi G T_{ij}, \quad i,j = 1,2,3, \quad i \neq j,$$

$$ds^2 = -dt^2 + a^2(t,\mathbf{x})(dx^2 + dy^2 + dz^2)$$

$$G_{00} = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\left(\frac{\nabla a}{a}\right)^2 - \frac{2}{a^2}\left(\frac{\nabla^2 a}{a}\right) = 8\pi G T_{00},$$
$$G_{ii} = -2a\ddot{a} - \dot{a}^2 - \left(\frac{\nabla a}{a}\right)^2 + \frac{\nabla^2 a}{a} + 2\left(\frac{\partial_i a}{a}\right)^2 - \frac{\partial_i^2 a}{a} = 8\pi G T_{ii},$$
$$G_{0i} = 2\frac{\dot{a}}{a}\frac{\partial_i a}{a} - 2\frac{\partial_i \dot{a}}{a} = 8\pi G T_{0i},$$
$$G_{ij} = 2\frac{\partial_i a}{a}\frac{\partial_j a}{a} - \frac{\partial_i \partial_j a}{a} = 8\pi G T_{ij}, \quad i,j = 1,2,3, \quad i \neq j,$$

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$$G_{ij} = 0 = 8\pi G T_{ij}, \quad i,j = 1,2,3, \quad i \neq j,$$

This leads to **completely different** **gravitational consequence for vacuum.**

$$ds^2 = -dt^2 + a^2(t,\mathbf{x})(dx^2 + dy^2 + dz^2)$$

$$G_{00} = 3\left(\frac{\dot a}{a}\right)^2 + \frac{1}{a^2}\left(\frac{\nabla a}{a}\right)^2 - \frac{2}{a^2}\left(\frac{\nabla^2 a}{a}\right) = 8\pi G T_{00},$$
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$$G_{ij} = 2\frac{\partial_i a}{a}\frac{\partial_j a}{a} - \frac{\partial_i \partial_j a}{a} = 8\pi G T_{ij}, \quad i,j = 1,2,3, \quad i \neq j,$$

First difference

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G T_{00} \quad \Rightarrow \quad 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{a^2}\left(\frac{\nabla a}{a}\right)^2 - \frac{2}{a^2}\left(\frac{\nabla^2 a}{a}\right) = 8\pi G T_{00}$$

Spatial derivatives of the scale factor $a(t, \mathbf{x})$ have to be large

$$2\frac{\partial_i a}{a}\frac{\partial_j a}{a} - \frac{\partial_i \partial_j a}{a} = 8\pi G T_{ij}, \qquad \langle T_{ij}\rangle = 0, \qquad \langle T_{ij}^2\rangle \sim \langle T_{00}\rangle^2. \qquad i \neq j$$

Vacuum energy density is **no longer** a direct **observable** quantity whose value is directly related to the **Hubble expansion rate**.

$$L(t) = a(t)\Delta x$$
$$\Delta x = |\mathbf{x}_1 - \mathbf{x}_2| \quad \Rightarrow \quad L(t) = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \sqrt{a^2(t, \mathbf{x})}dl$$

$$H(t) = \frac{\dot{L}}{L} = \frac{\dot{a}}{a} = \pm\sqrt{\frac{8\pi G T_{00}}{3}} \quad \Rightarrow \quad H(t) = \frac{\dot{L}}{L} = \frac{\int_{\mathbf{x}_1}^{\mathbf{x}_2} \frac{\dot{a}}{a}(t, \mathbf{x})\sqrt{a^2(t, \mathbf{x})}dl}{\int_{\mathbf{x}_1}^{\mathbf{x}_2} \sqrt{a^2(t, \mathbf{x})}dl}$$

Second difference

The **local Hubble rates** $\frac{\dot{a}}{a}(t, \mathbf{x})$ have to be constantly changing signs over space and time

$$2\frac{\dot{a}}{a}\frac{\partial_i a}{a} - 2\frac{\partial_i \dot{a}}{a} = 8\pi G T_{0i} \quad \Longleftrightarrow \quad \nabla\left(\frac{\dot{a}}{a}\right) = -4\pi G \mathbf{J}, \quad \mathbf{J} = (T_{01}, T_{02}, T_{03})$$

Initial value constraints on local Hubble rates

$$\frac{\dot{a}}{a}(t, \mathbf{x}) = \frac{\dot{a}}{a}(t, \mathbf{x}_0) - 4\pi G \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{J}(t, \mathbf{x}') \cdot \mathbf{dl}'$$

Due to huge fluctuations of the energy flux

$$\langle \mathbf{J} \rangle = \mathbf{0} \qquad J = \sqrt{\langle \mathbf{J}^2 \rangle} \sim \langle T_{00} \rangle \sim \Lambda^4 \to +\infty$$

Difference in local Hubble rates becomes comparable to itself for points separated by only a distance of the order $\Delta x \sim \frac{1}{\sqrt{G}\Lambda^2}$ $\quad \Lambda \to +\infty$

$$\sqrt{\left\langle \left(\frac{\dot{a}}{a}\right)^2 \right\rangle} \sim \sqrt{G \langle T_{00} \rangle} \sim \sqrt{G}\Lambda^2 \qquad \Delta\left(\frac{\dot{a}}{a}\right) \sim 4\pi G J \Delta x \sim \sqrt{G}\Lambda^2 \sim \sqrt{\left\langle \left(\frac{\dot{a}}{a}\right)^2 \right\rangle}$$

Third difference

The linear combination $G_{00} + \frac{1}{a^2}(G_{11} + G_{22} + G_{33}) = -\frac{6\ddot{a}}{a}$

(all the spatial derivatives of $a(t, \mathbf{x})$ are cancelled)

gives the **dynamic equation of motion** for the scale factor $a(t, \mathbf{x})$

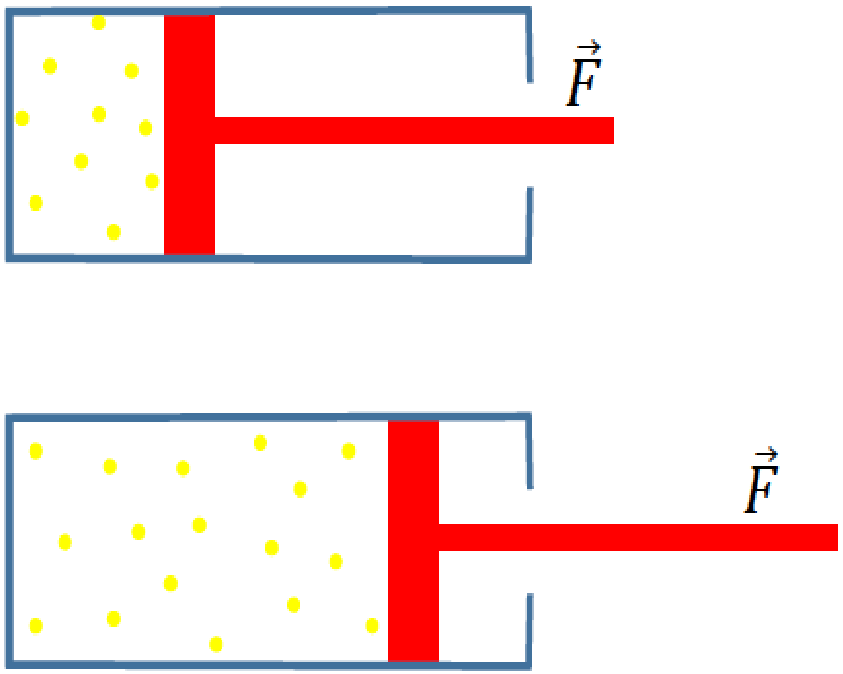
$$\ddot{a} + \Omega^2(t, \mathbf{x})a = 0,$$

where $\Omega^2 = \frac{4\pi G}{3}\left(\rho + \sum_{i=1}^{3} P_i\right),$ $\rho = T_{00},\ P_i = \frac{1}{a^2}T_{ii}.$

This equation is a generalization of the **second Friedmann equation**

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

The sign of Ω^2



ρ=*constant*
$P = -\rho$

If treating ρ as a constant,

we must have $P = -\rho$

as required by the conservation equation

$$\nabla^{\mu} T_{\mu\nu} = 0$$

Then

$$\Omega^2 = \frac{4\pi G}{3}(\rho + 3P) = -\frac{8\pi G\rho}{3} < 0$$

“Repulsive” gravity

If ρ is not a constant, for example, consider contribution to Ω^2

from a massless scalar field ϕ : $T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\lambda \phi \nabla_\lambda \phi$

Then the conservation equation $\nabla^\mu T_{\mu\nu} = 0$ is automatically satisfied

And $\rho + \sum_{i=1}^{3} P_i = 2\dot{\phi}^2$ $\Omega^2 = \frac{8\pi G \dot{\phi}^2}{3} > 0$ **"Attractive" gravity**

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And $\rho + \sum_{i=1}^{3} P_i = 2\dot{\phi}^2$ $\Omega^2 = \frac{8\pi G \dot{\phi}^2}{3} > 0$ **"Attractive" gravity**

The **dynamic equation of motion**

$$\ddot{a} + \Omega^2(t, \mathbf{x}) a = 0,$$

describes a **harmonic oscillator** with **time dependent** frequency.

Basic property: it **oscillates back and forth** results in **huge cancellations** in **observed Hubble expansion rates**.

$$H(t) = \frac{\dot{L}}{L} = \frac{\int_{\mathbf{x}_1}^{\mathbf{x}_2} \frac{\dot{a}}{a}(t, \mathbf{x}) \sqrt{a^2(t, \mathbf{x})} dl}{\int_{\mathbf{x}_1}^{\mathbf{x}_2} \sqrt{a^2(t, \mathbf{x})} dl}$$

Adiabatic evolution

Key equation $\ddot{a} + \Omega^2(t, \mathbf{x})a = 0,$

The oscillation frequency of the harmonic oscillator

$$\Omega \sim \sqrt{G \langle T_{00} \rangle} \sim \sqrt{G}\Lambda^2$$

(Dimensional analysis)

From the energy-time uncertainty principle

$$\Delta E \Delta t \sim 1, \ \Delta E \sim \Lambda$$

Ω has a significant change after the time scale

$$\Delta t \sim 1/\Lambda$$

Ω is **slowly varying** that this is basically an **adiabatic process**

$$T = 2\pi/\Omega \sim 1/\sqrt{G}\Lambda^2 \ll 1/\Lambda, \text{ as } \Lambda \to \infty$$

Leading order solution (WKB)

$$a(t,\mathbf{x}) = \frac{A_0}{\sqrt{\Omega(t,\mathbf{x})}} \cos\left(\int_0^t \Omega(t',\mathbf{x})dt' + \theta_{\mathbf{x}}\right)$$

The two constants A_0 and $\theta_{\mathbf{x}}$ depends on
the initial values $a(0,\mathbf{x})$ and $\dot{a}(0,\mathbf{x})$

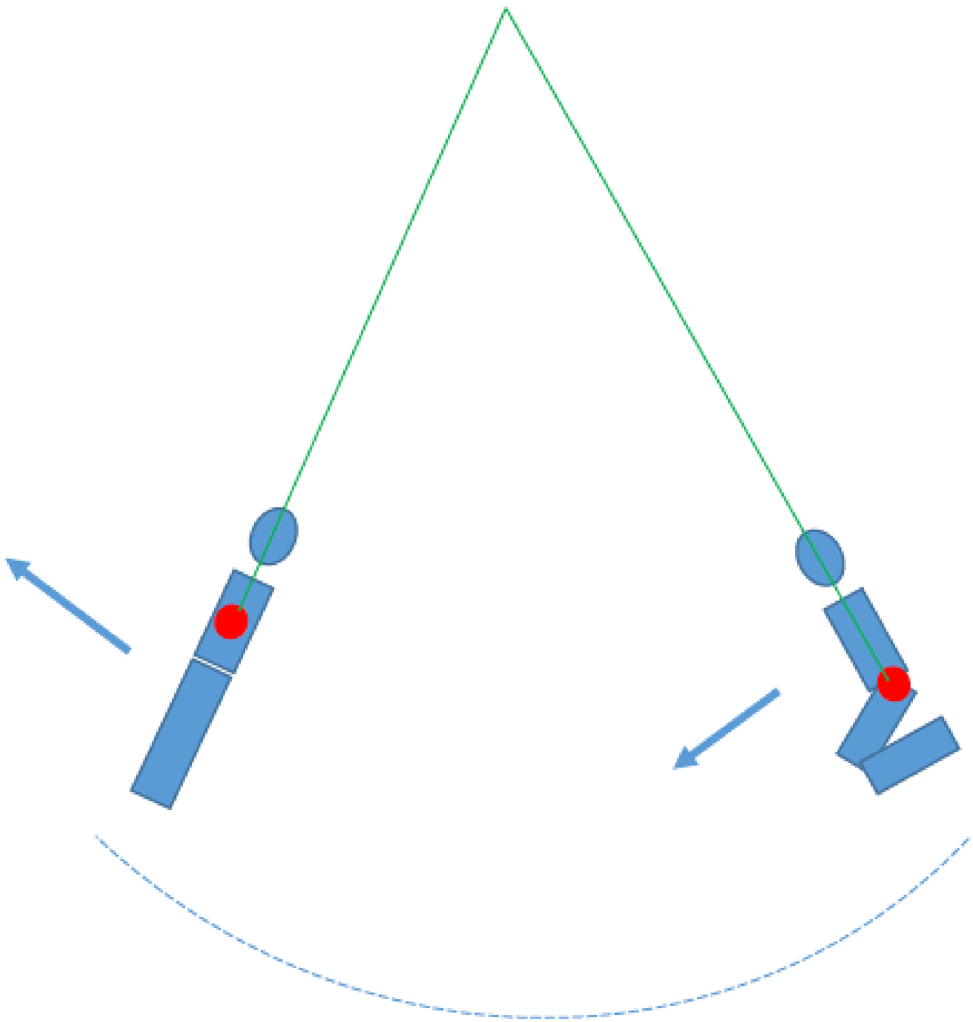
The initial phase $\theta_{\mathbf{x}}$ can be determined by the constraint

$$\frac{\dot{a}}{a}(t,\mathbf{x}) = \frac{\dot{a}}{a}(t,\mathbf{x}_0) - 4\pi G \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{J}(t,\mathbf{x}') \cdot d\mathbf{l}' \qquad \mathbf{J} = (T_{01}, T_{02}, T_{03})$$

It gives

$$\tan\theta_{\mathbf{x}} = \frac{\Omega(0,\mathbf{x}_0)}{\Omega(0,\mathbf{x})} \tan\theta_{\mathbf{x}_0} + \frac{4\pi G}{\Omega(0,\mathbf{x})} \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{J}(0,\mathbf{x}') \cdot d\mathbf{l}'$$

Accelerating expansion from weak parametric resonance



Parametric resonance

$$\ddot{a} + \Omega^2(t, \mathbf{x})a = 0, \qquad \Omega = \sqrt{8\pi G\dot{\phi}^2/3}$$

If $\Omega(t, \mathbf{x})$ is strictly periodic in time with a period T (Floquet theory)

$$a(t, \mathbf{x}) = c_1 e^{H_{\mathbf{x}} t} P_1(t, \mathbf{x}) + c_2 e^{-H_{\mathbf{x}} t} P_2(t, \mathbf{x}) \qquad H_{\mathbf{x}} > 0$$

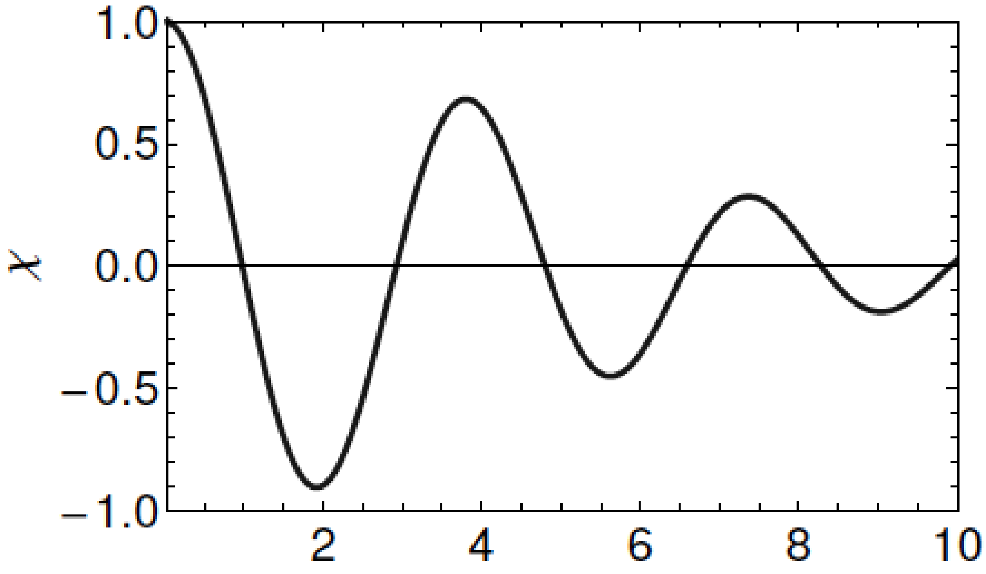
First term dominant $\quad a(t, \mathbf{x}) \simeq e^{H_{\mathbf{x}} t} P(t, \mathbf{x})$

P_1 P_2 P are periodic functions with the period T .

Physically, after each period of evolution, a increases by a fixed ratio

$$a(t+T, \mathbf{x}) = \mu_{\mathbf{x}} a(t, \mathbf{x}) \qquad a(t+nT, \mathbf{x}) = \mu_{\mathbf{x}}^n a(t, \mathbf{x}) \qquad H_{\mathbf{x}} = \frac{\ln \mu_{\mathbf{x}}}{T}$$

$\Omega(t, \mathrm{x})$ is not strictly periodic but **quasi-periodic**



$$\chi(\Delta t) = \mathrm{Cov}\left(\Omega^2(t_1, \mathrm{x}), \Omega^2(t_2, \mathrm{x})\right)$$
$$= \frac{\left\langle \left\{ \left(\Omega^2(t_1) - \left\langle \Omega^2(t_1) \right\rangle\right) \left(\Omega^2(t_2) - \left\langle \Omega^2(t_2) \right\rangle\right) \right\} \right\rangle}{\left\langle \left(\Omega^2 - \langle \Omega^2 \rangle\right)^2 \right\rangle}$$

$\Omega(t, \mathrm{x})$ varies around its mean value **quasiperiodically** on the time scale $T \sim 1/\Lambda$.

Due to the quasiperiodicity of $\Omega(t, \mathrm{x})$, the system exhibits similar **parametric resonance** behavior.

During the ith cycle of time duration T_i , $a(t+T_i, \mathrm{x}) = \mu_{i\mathrm{x}} a(t, \mathrm{x})$

After n cycles $$a(t+\sum_{i=1}^{n} T_i, \mathrm{x}) = \left(\prod_{i=1}^{n} \mu_{i\mathrm{x}}\right) a(t, \mathrm{x})$$

The evolution of $a(t, \mathrm{x})$ is quasi-exponential

$$a(t, \mathrm{x}) \simeq e^{\int_0^t H_{\mathrm{x}}(t')dt'} P(t, \mathrm{x})$$

$P(t, \mathrm{x})$ is a quasiperiod function with the same quasiperiod of the order $1/\Lambda$ as the time dependent frequency $\Omega(t, \mathrm{x})$

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Physical length $$L(t) = L(0)e^{Ht}$$

where $$L(0) = \int_{\mathrm{x}_1}^{\mathrm{x}_2} \sqrt{P^2(t, \mathrm{x})} dl \qquad H = \frac{1}{t}\int_0^t H_{\mathrm{x}}(t')dt'$$

The **accelerating expansion** from a different mechanism!

$$\ddot{x} + \omega^2(t)x = 0 \qquad \omega^2(t) = \omega_0^2\,(1 + h\cos\gamma t)$$

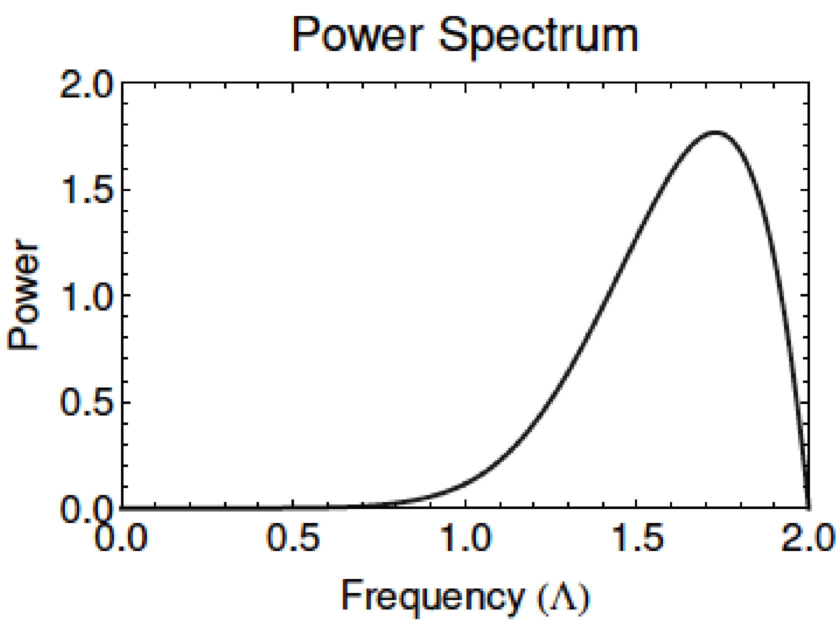
Parametric resonance occurs if $\gamma \sim \frac{2\omega_0}{n}$. As $n \to \infty$,

the strength of the parametric resonance **decreases to zero**.

$$\ddot{a} + \Omega^2(t, \mathbf{x})a = 0, \qquad \Omega^2 = 8\pi G\dot{\phi}^2/3$$

$$\Omega^2(t, \mathbf{0}) = \Omega_0^2\left(1 + \int_0^{2\Lambda} d\gamma\,(f(\gamma)\cos\gamma t + g(\gamma)\sin\gamma t)\right)$$

$$\Omega_0^2 = \langle\Omega^2\rangle = \frac{G\Lambda^4}{6\pi}$$



$$\Omega_0 \sim \sqrt{G}\Lambda^2 \gg \quad 2\Lambda$$

Always exists

$$n \geq \sqrt{\frac{G}{6\pi}}\Lambda, \quad \Lambda \to +\infty,$$

such that $\frac{2\Omega_0}{n} \in (0, 2\Lambda)$

Therefore, the global Hubble expansion rate

$$H \to 0, \qquad \text{as} \qquad \Lambda \to +\infty$$

This is **opposite** to the usual prediction made
by treating the vacuum energy density ρ^{vac} as a constant.

$$H = \sqrt{8\pi G \rho^{vac}/3} \propto \sqrt{G}\Lambda^2 \to \infty \qquad \text{as} \qquad \Lambda \to +\infty$$

So the gravitational consequence of the huge vacuum energy density predicted by applying QM and GR is not huge but small.

Estimation of the dependence of the Hubble rate H on Λ

The adiabatic solution does not include the weak parametric resonance effect. This effect will lead to

$$A_0 \Longrightarrow A(t,\mathbf{x}) = A_0 e^{\int_0^t H_{\mathbf{x}}(t')dt'}$$

To determine $H_{\mathrm{x}}(t)$, consider the adiabatic invariant

$$I(t,\mathrm{x}) = E/\Omega = \frac{1}{2}(\dot{a}^2 + \Omega^2 a^2)/\Omega = \frac{1}{2}A^2(t,\mathrm{x})$$

Do the canonical transformation leads to

$$a = \sqrt{2I/\Omega}\sin\varphi, \quad \dot{a} = \sqrt{2I\Omega}\cos\varphi. \Longrightarrow \frac{dI}{dt} = -I\frac{\dot{\Omega}}{\Omega}\cos 2\varphi, \quad \frac{d\varphi}{dt} = \Omega + \frac{\dot{\Omega}}{2\Omega}\sin 2\varphi.$$

Evolution of the adiabatic invariant

$$I(t) = I(0)\exp\left(2\int_0^t H_{\mathbf{x}}(t')dt'\right)$$

where $$H_{\mathbf{x}}(t') = -\frac{\dot{\Omega}}{2\Omega}\cos 2\varphi$$

The global Hubble expansion rate

$$H = \mathrm{Re}\left(-\frac{1}{t}\int_0^t \frac{\dot{\Omega}}{2\Omega}e^{2i\varphi}dt'\right)$$

Change the integration variable from t' to φ'

$$H = \mathrm{Re}\left(-\frac{1}{t}\int_{\varphi_0}^{\varphi} \frac{\dot{\Omega}}{2\Omega}e^{2i\varphi}\frac{dt'}{d\varphi'}d\varphi'\right)$$

where $\varphi_0 = \varphi(0)$ $\qquad \varphi = \varphi(t)$

To evaluate H, we formally treat φ as a complex variable and close the contour integral in the upper half plane.

Since $\varphi \sim \Omega t \sim \sqrt{G}\Lambda^2 t$, the interval of integration

$$\varphi - \varphi_0 \sim \sqrt{G}\Lambda^2 t \quad \to +\infty \qquad \text{as} \qquad \Lambda \to +\infty$$

The principle contribution to H comes from the residue values at singularities $\varphi_{(k)}$ inside the contour

$$H = \frac{1}{t}\,\mathrm{Re}\left(2\pi i \sum_k \mathrm{Res}\left(-\frac{\dot{\Omega}}{2\Omega} e^{2i\varphi}\frac{dt}{d\varphi},\ \varphi_{(k)}\right)\right)$$

Each term gives a contribution containing a factor

$$\exp\left(-2\,\mathrm{Im}\,\varphi_{(k)}\right)$$

Dominant contribution to H comes from the singularities near the real axis, i.e. those with smallest real positive imaginary part.

$\Omega(t)$ varies quasiperiodically with a characteristic time $\tau \sim 1/\Lambda$

The number of singularities near the real axis would be roughly on the order $t/\tau \sim \Lambda t$ and thus H is roughly

$$H \sim \Lambda \exp\left(-2\,\mathrm{Im}\,\varphi_{(k)}\right)$$

Let $t_{(k)}$ be the (complex) "instant" corresponding to the singularity $\varphi_{(k)}$: $\varphi_{(k)} = \varphi(t_{(k)}) \sim \Omega\, t_{(k)}$

In general, $|t_{(k)}|$ has the same order of magnitude as the characteristic time $\tau \sim 1/\Lambda$ of variation of $\Omega(t)$.
Remember that $\Omega \sim \sqrt{G}\Lambda^2$, we have

$$\mathrm{Im}\,\varphi_{(k)} \sim \Omega\tau \sim \sqrt{G}\Lambda$$

Therefore, we obtain that

$$H = \alpha\Lambda e^{-\beta\sqrt{G}\Lambda}$$

α and β are two dimensionless constants whose values depend on the variation details of the time dependent frequency.

Numerical verification

Weyl transformation of Ω^2

$$\Omega^2(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t) = \frac{8\pi}{3}\int \frac{d^3k d^3k'}{(2\pi)^3} x_{\mathbf{k}} x_{\mathbf{k}'} \omega\omega' \sin\omega t \sin\omega' t$$
$$+ p_{\mathbf{k}} p_{\mathbf{k}'} \cos\omega t \cos\omega' t - 2 x_{\mathbf{k}} p_{\mathbf{k}'} \omega \sin\omega t \cos\omega' t.$$

For a particular choice of $\{x_{\mathbf{k}}, p_{\mathbf{k}}\}$, we have a classic equation for a

$$\ddot{a}(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t) + \Omega^2(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t)a(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t) = 0$$

Observed value $a_o(t)$ is the average of $a(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t)$ over the Wigner pseudo distribution function $W(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t)$

$$a_o(t) = \int \left(\prod_{\mathbf{k}} dx_{\mathbf{k}} dp_{\mathbf{k}}\right) a(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t) W(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t)$$

$$W(\{x_{\mathbf{k}}\},\{p_{\mathbf{k}}\},t) = \prod_{\mathbf{k}} \frac{1}{\pi} e^{-\frac{p_{\mathbf{k}}^2}{\omega} - x_{\mathbf{k}}^2 \omega}$$

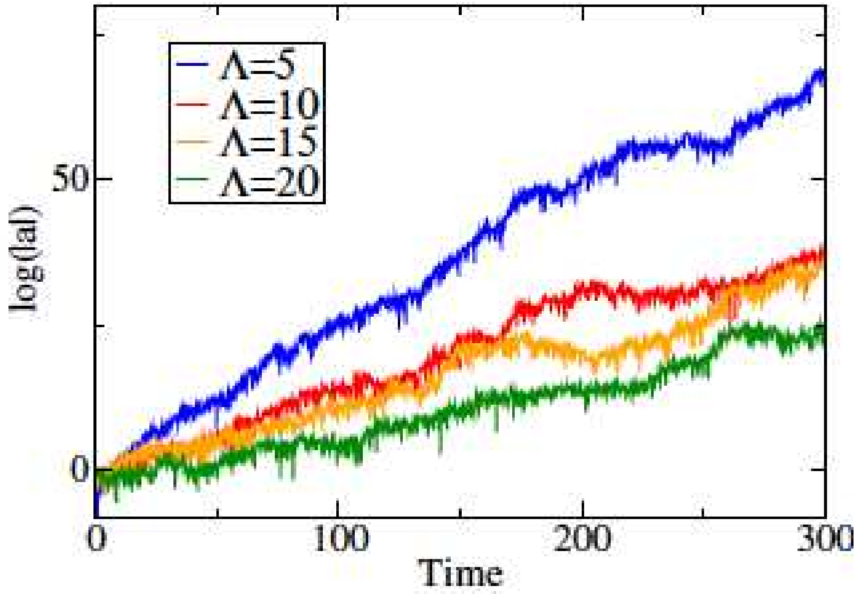


FIG. 4. Numeric result for $\log|a_o(t)|$ for a single real massless scalar field for different Λ.

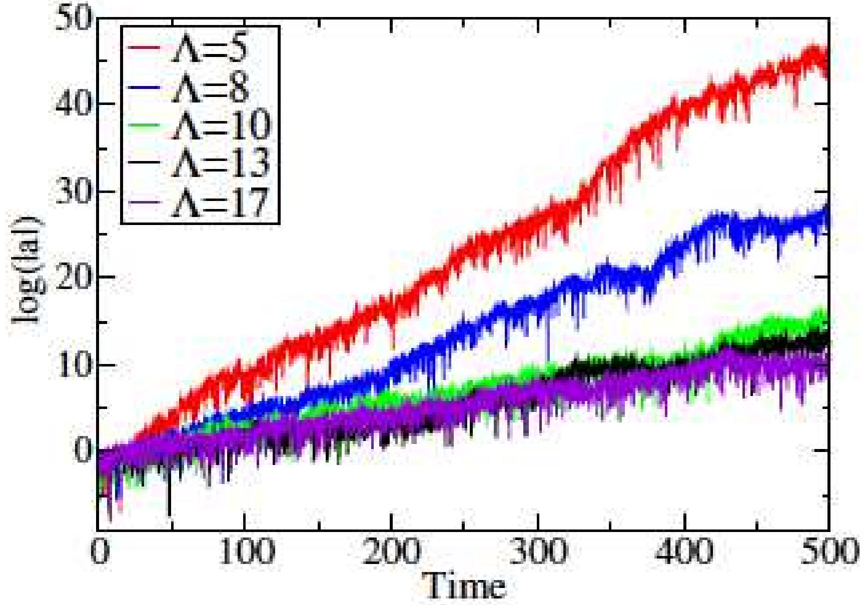


FIG. 5. Numeric result for $\log|a_o(t)|$ when two Klein-Gordon fields are present and we do discover that as Λ increases, the slope of $\log|a_o(t)|$ starts to decreases.

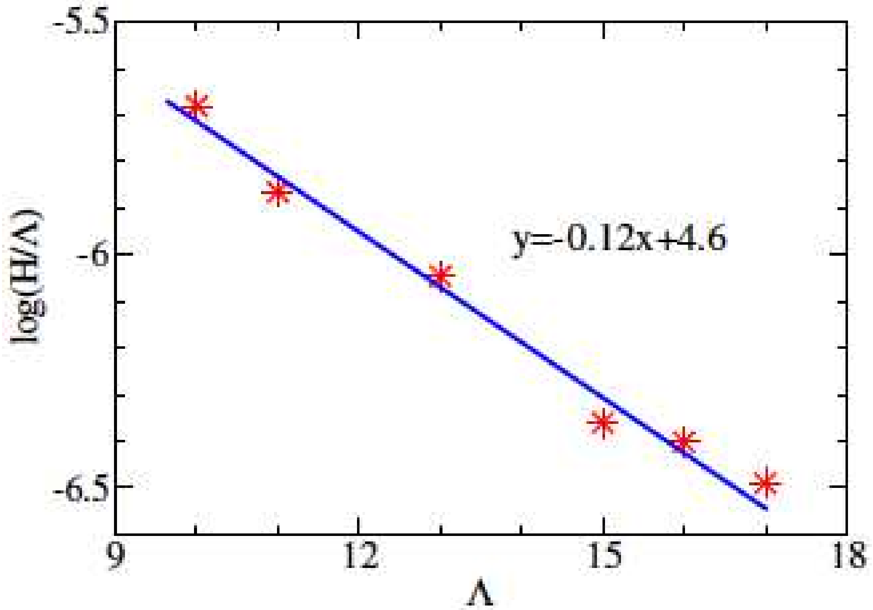


FIG. 6. The plot of $\log(H/\Lambda)$ over Λ. The fitting result shows that $\alpha = e^{4.6} \approx 100$ and $\beta = 0.12$ in this two-field case.

Here we further set G=1 and using Planck units

$$H = \alpha e^{-\beta\Lambda}$$

$$\log(H/\Lambda) = -\beta\Lambda + \log\alpha$$

Meaning of the results

$$H = \sqrt{\frac{8\pi G \rho^{\text{vac}}}{3}} \propto \sqrt{G}\Lambda^2 \to +\infty$$

$$H = \alpha \Lambda e^{-\beta \sqrt{G} \Lambda} \to 0$$

In this sense, the old cosmological constant problem would be resolved.

Always exists a large value for the cutoff to match the current observed rate of the accelerating expansion of the Universe.

For two scalar fields, $\Lambda \sim 1000 E_P$

More fields will reduce the energy of cutoff needed.

More general inhomogeneous coordinates

$$ds^2 = -dt^2 + h_{ab}(t,\mathbf{x})dx^a dx^b, \quad a,b = 1,2,3.$$

Six evolution equation for the second fundamental form

$$\dot{k}_{ab} = -R^{(3)}_{ab} - (trk)k_{ab} + 2k_{ac}k^c_b$$
$$+4\pi G\rho h_{ab} + 8\pi G\left(T_{ab} - \frac{1}{2}h_{ab}trT\right)$$

Four constraint equation

$$R^{(3)} + (trk)^2 - k_{ab}k^{ab} = 16\pi G\rho,$$

$$D_a k^a_b - D_b(trk) = 8\pi G j_b,$$

where

$$k_{ab} = \frac{1}{2}\dot{h}_{ab}, \; k^{ab} = h^{ac}h^{bd}k_{cd}, \; trk = h^{ab}k_{ab},$$

$$\rho = T_{00}, \; j_b = h^a_b T_{0a}, \; trT = h^{ab}T_{ab}$$

Taking trace on both sides of the six evolution equation and then combine with the first constraint equation gives

$$h^{ab}\dot{k}_{ab} - k_{ab}k^{ab} = -4\pi G\,(\rho + trT)$$

All spatial derivatives are still cancelled.

$\ddot{a} + \Omega^2(t, \mathrm{x})a = 0$ is a special case of the above equation

For massless scalar field

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\phi\nabla_\lambda\phi$$

We still have

$$\rho + trT = 2\dot{\phi}^2$$

All spatial derivatives of ϕ are also cancelled and still no explicit dependence on the metric $g_{\mu\nu}$.

Consider the following special case

$$h_{ab}(t,\mathbf{x}) = \begin{pmatrix} a^2(t,\mathbf{x}) & 0 & 0 \\ 0 & b^2(t,\mathbf{x}) & 0 \\ 0 & 0 & c^2(t,\mathbf{x}) \end{pmatrix}$$

More freedoms and more rich structures

Local expansion rates $\dot{a}/a,\ \dot{b}/b$ and $\dot{c}/c$ may have different phases along the three eigen-directions $\hat{x},\ \hat{y}$ and $\hat{z}$.

Evolution equation becomes

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -4\pi G\left(\rho + trT\right)$$

Let $$\frac{\ddot{a}}{a} = -\Omega_1^2(t,\mathbf{x}), \quad \frac{\ddot{b}}{b} = -\Omega_2^2(t,\mathbf{x}), \quad \frac{\ddot{c}}{c} = -\Omega_3^2(t,\mathbf{x}),$$

Then we have $$\Omega_1^2(t,\mathbf{x}) + \Omega_2^2(t,\mathbf{x}) + \Omega_3^2(t,\mathbf{x}) = 4\pi G\left(\rho + trT\right)$$

$$\left\langle \Omega_i^2(t,\mathbf{x}) \right\rangle = \frac{4\pi G}{3}\left\langle \rho + trT \right\rangle, \quad i = 1,2,3.$$

Ω_i^2 must also be **slow varying** functions.

Similarly, we have

$$a(t,\mathbf{x}) \simeq e^{\int_0^t H_{1\mathbf{x}}(t')dt'} P_1(t,\mathbf{x}),$$
$$b(t,\mathbf{x}) \simeq e^{\int_0^t H_{2\mathbf{x}}(t')dt'} P_2(t,\mathbf{x}),$$
$$c(t,\mathbf{x}) \simeq e^{\int_0^t H_{3\mathbf{x}}(t')dt'} P_3(t,\mathbf{x}),$$

And the global Hubble expansion rate in ith direction

$$H_i = \frac{1}{t}\int_0^t H_{i\mathbf{x}}(t')dt' \qquad H_i = \alpha\Lambda e^{-\beta\sqrt{G}\Lambda}, \quad i = 1, 2, 3,$$

The observed physical volume

$$V(t) = \int \sqrt{h(t,\mathbf{x})}d^3x = V(0)e^{3Ht}$$

where $$h = \det h_{ab} = a^2b^2c^2$$

For the most general case

$$h_{ab}(t,\mathbf{x}) = \begin{pmatrix} a^2(t,\mathbf{x}) & d(t,\mathbf{x}) & e(t,\mathbf{x}) \\ d(t,\mathbf{x}) & b^2(t,\mathbf{x}) & f(t,\mathbf{x}) \\ e(t,\mathbf{x}) & f(t,\mathbf{x}) & c^2(t,\mathbf{x}) \end{pmatrix}$$

The three **orthogonal eigenvectors** of the symmetric matrix h_{ab} can **rotate** in space.

An initial sphere will distort toward an ellipsoid with principle axes given by **eigenvectors** of h_{ab}, with rates given by time derivatives $\dot{\lambda}_i/\lambda_i$ of the corresponding **eigenvalues** $\lambda_i^2(t,\mathbf{x}), i = 1,2,3.$

Suggestion from previous results: the **eigenvalues** $\lambda_i^2(t,\mathbf{x})$ should also evolve **adiabatically** similar to a^2, b^2 and c^2, then we expect

$$\begin{aligned} V(t) &= \int \sqrt{h(t,\mathbf{x})}\, d^3x \\ &= \int \sqrt{\lambda_1^2\lambda_2^2\lambda_3^2}\, d^3x \\ &= V(0)e^{3Ht}, \end{aligned}$$

where

$$h = \det(h_{ab})$$

$$H = \alpha\Lambda e^{-\beta\sqrt{G}\Lambda}$$

Conclusion

- The gravitational effect produced by the huge vacuum energy density is still huge, but confined to Planck scales. Only a small observable net effect left on the cosmological scale-----the accelerating expansion of the Universe.

- This physical picture looks crazy at first glance, but it is just the prediction of Quantum Mechanics and General Relativity.