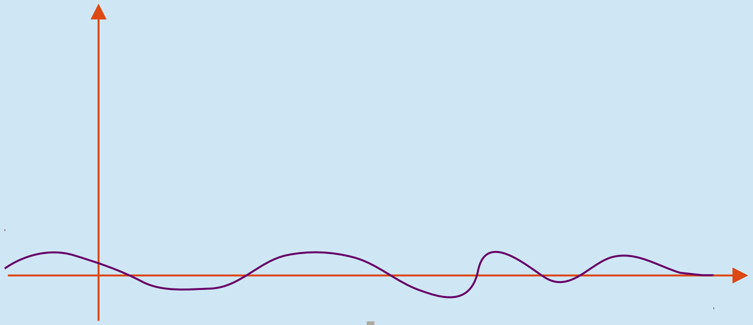
Gravitational collapse of dynamically cold systems

Description of the problem

We start from initially cold conditions



What is going to be the final “equilibrium” state ?

Statistical approach

Theory of violent relaxation

Lynden Bell, 1967, MNRAS, 136, 101

Each particle is subject to a violent relaxation process
As a result the memory of its initial conditions is erased
Each cell of constant volume in phase space as the same probability

Nakamura, 2000, 731, 739
Violent relaxation too but each cell of constant mass as equal probability

Numerical testing of these two theories

Itai Arad , 2005, MNRAS, 362 252

Result: none of them work in practice....

How to characterize the outcome of gravitational collapse ?

The problem of the infall of matter on a point mass is scale free

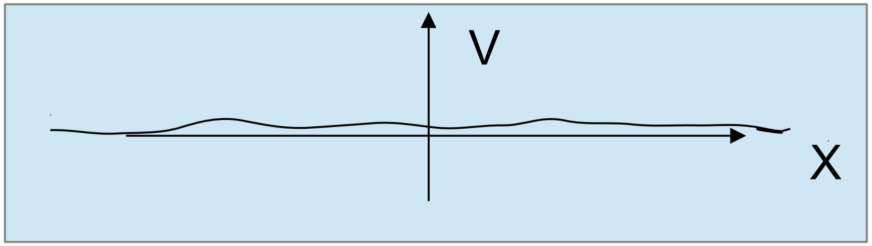
This symmetry should be apparent in the final state

This final solution is not necessarily stationary
But look stationary using some proper smoothing of the solution fine details

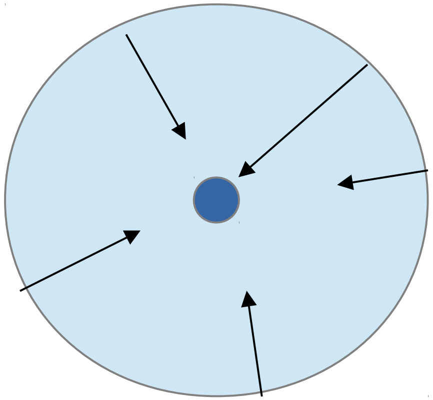
Some mysterious self similar/scale free properties appear
in numerical simulations of the collapse of CDM haloes
Taylor & Navarro, 2001, ApJ, 563, 483

The Cosmological infall induce self similarity in the solution

The dark matter in the initial conditions is dynamically cold



The infall of this dark matter on a point mass
Has self similar properties
(all time dependencies may be re-scaled by a power on time)



Self similar solutions in phase space

No dependence on scale for the phase space density solution F

$$f\left(\lambda_1 \boldsymbol{x}, \lambda_2 \boldsymbol{v}, \lambda_3 t\right)=\lambda_4 f\left(\boldsymbol{x}, \boldsymbol{v}, t\right)$$

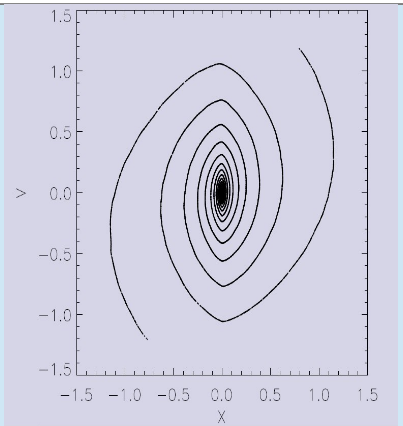
Infinitesimal variations of λ implies solution for F

$$f\left(\boldsymbol{x}, \boldsymbol{v}, t\right)=t^{\alpha_0} F\left(\frac{\boldsymbol{x}}{t^{\alpha_1}}, \frac{\boldsymbol{v}}{t^{\alpha_2}}\right)$$

Short summary

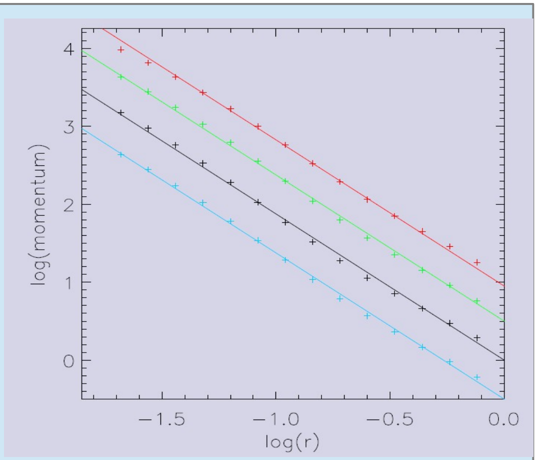
I

Analytical equation for the spiral in phase space
(Power law force field)
Prediction of geometry 1D & 3D central region



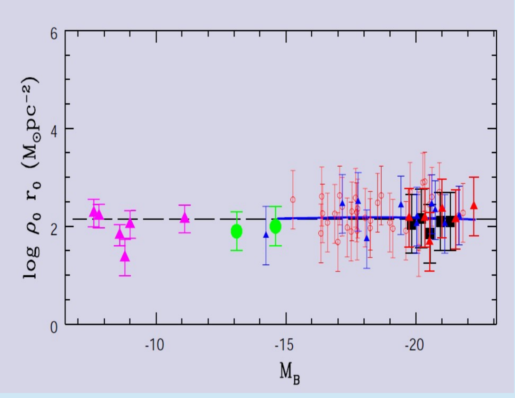
II

Statistical properties of smoothed solutions: collapse in numerical
Simulations of DM is fully self similar.
A general test of self similarity up to the Nth order.



III

Consequence of the self similar framework when the baryonic
Feedback is introduced: new auto similarity class
Observational phenomenology is connected and integrated
In a consistent framework.



The n-dimensional Vlasov equation :

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \boldsymbol{x}}\boldsymbol{v}-\frac{\partial \Phi}{\partial \boldsymbol{x}}\frac{\partial f}{\partial \boldsymbol{v}}=0$$

The same equation in reduced variables :

$$(2+n\alpha_2)F+(1+\alpha_2)\frac{\partial F}{\partial \boldsymbol{x_2}}\boldsymbol{x_2}+\alpha_2\frac{\partial F}{\partial \boldsymbol{v_2}}-\frac{\partial F}{\partial \boldsymbol{x_2}}\boldsymbol{v_2}+\frac{\partial \Phi}{\partial \boldsymbol{x_2}}\frac{\partial F}{\partial \boldsymbol{v_2}}=0 \qquad (1)$$

$$\alpha_0=-2-n\alpha_2 \qquad \alpha_1=1+\alpha_2$$

$$\boldsymbol{x_2}=\frac{\boldsymbol{x}}{t_1^{\alpha}} \qquad \boldsymbol{v_2}=\frac{\boldsymbol{v}}{t_1^{\alpha}}$$

How to find a solution to Eq (1) ?

Self similarity is a simplification and a reduction of the general problem,
But Eq (1) is still very complex

Is finding a solution hopeless in itself ?

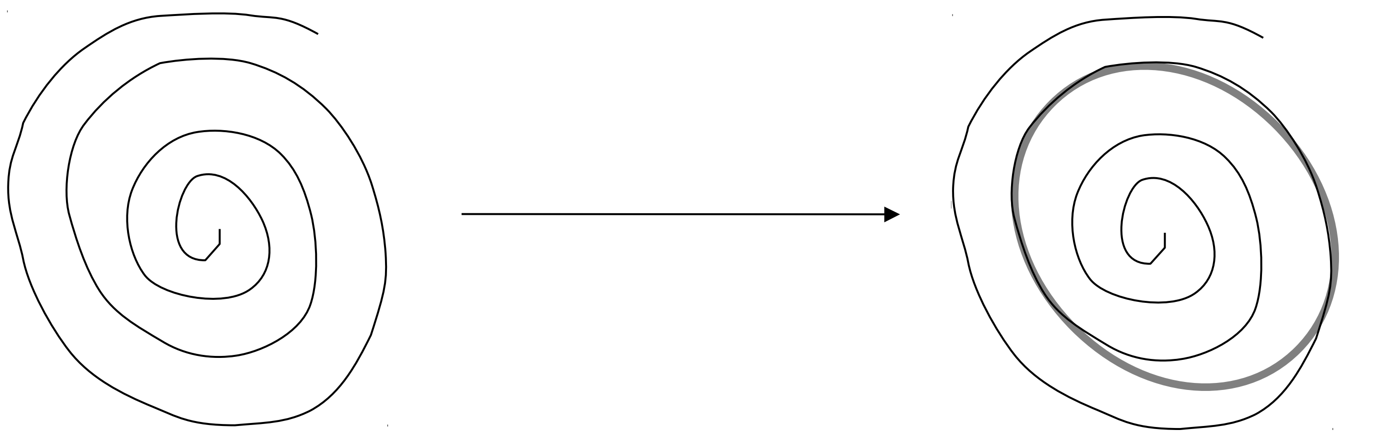
Some clues on the solution :

It is a series of folds in phase space
We consider an adiabatic regime

The clues

1) Series of folds :

Adiabatic



Need some kind of polar coordinates

The potential variations are small
During an orbit : thus a fold is
close to a line of constant energy

Let find the solution for a general power law potential

$$\Phi = k\,x^{\beta+2} \qquad E = k\,x^{\beta+2} + \frac{v^2}{2}$$

Equation of a spiral, $h(R,\theta) = c_0$ a fold implies $\theta = \theta + 2\pi$

If θ is large the angular variation during a fold is small and the fold $\theta = \theta + 2\pi$
Corresponds to R=Constant=E

Constant density, k=1/2, $E = \frac{1}{2}(x^2 + v^2)$

Idenfication with polar coordinates, $R^2 = E$, $\tan(\theta) = \frac{v}{x}$

General case : change variable, $u = x^{\eta}$ → $E = u^2 + v^2$

Introducing the appropriate change in variable

$u=|x_2|^{\eta}$

$\eta=\frac{\beta}{2}+1$

$G=\ln(F)$

$$2+\alpha_2+(1+\alpha_2)\eta\frac{\partial G}{\partial u}u+\alpha_2\frac{\partial G}{\partial v_2}v_2-sgn(x_2)\eta\left(\frac{\partial G}{\partial u}v_2-u\frac{\partial G}{\partial v_2}\right)u^{\frac{\beta}{\beta+2}}=0$$

Polar coordinates:

$$R=\sqrt{u^2+v_2^2}$$

$$\tan(\psi)=\frac{v}{u}$$

$$-\eta_2\sin\psi\cos\psi(1+\alpha_2)\frac{\partial H}{\partial\psi}+R\left(\cos\psi^2(1+\alpha_2)\eta_2+\alpha_2\right)\frac{\partial H}{\partial R}+\eta\frac{\partial H}{\partial\psi}\left(R|\cos\psi|\right)^{\frac{\beta}{\beta+2}}+\alpha_2+2=0$$

$\eta_2=\eta-\frac{\alpha_2}{\alpha_2+1}$

The nearly adiabatic regime: 2 interesting cases

1) Stationary solution
2) very close to origin=large number of folds

Stationary solution

$$\rho(t) = t^{-2}\rho_2(t) = t^{-2}x_2^{\beta} = t^{-2}\left(\frac{x_2}{t^{\alpha_1}}\right)^{\beta} \rightarrow \beta\alpha_1 = -2 \rightarrow \eta_2 = 0$$

The Vlasov self similar equation is transformed to:

$$R\alpha_2\frac{\partial H}{\partial R} + \eta\frac{\partial H}{\partial\psi}\left(R\left|\cos\psi\right|\right)^{\frac{\beta}{\beta+2}} + \alpha_2 + 2 = 0$$

This equation has a general solution

$$H(R,\psi) = -\frac{\alpha_2+2}{\alpha_2}\log R + Q\left(R^{-\frac{1}{\alpha_2}} + \frac{1+\alpha_2}{\alpha_2}\int\left|\cos\psi\right|^{\frac{1}{\alpha_2}}d\psi\right)$$

The iso contours of the solution are spirals

Small radius approximation

Equation is reduced to dominant terms:

$$\eta \frac{\partial H}{\partial \psi} \left(R \left| \cos \psi \right| \right)^{\frac{\beta}{\beta+2}} + \alpha_2 + 2 = 0$$

With general solution:

$$H(R,\psi) = -\frac{(1+\alpha_2)(2+\alpha_2)}{\alpha_2} R^{\frac{1}{\alpha_2}} \int \left| \cos \psi \right|^{\frac{1}{\alpha_2}} d\psi + H_2(R)$$

The iso-contours are also spirals consistent with the stationary case

Solution iso-contours: geometry of the spirals

$$\begin{cases} \tilde{Q}\left(R^{-\frac{1}{\alpha_2}}+I(\psi)\right)=R^{\frac{\alpha_2+2}{\alpha_2}}F_0 & \longrightarrow \text{Eq. of the iso-contour} \\ Q=\log\tilde{Q} \\ I(\psi)=(1+\alpha_2)\int|\cos\psi|^{\frac{1}{\alpha_2}}d\psi \end{cases}$$

A fold implies:

$$\begin{cases} \psi=\psi_0+2\pi=\psi_0+\epsilon\, d\psi \\ R=R_0+\epsilon\, dR \end{cases}$$

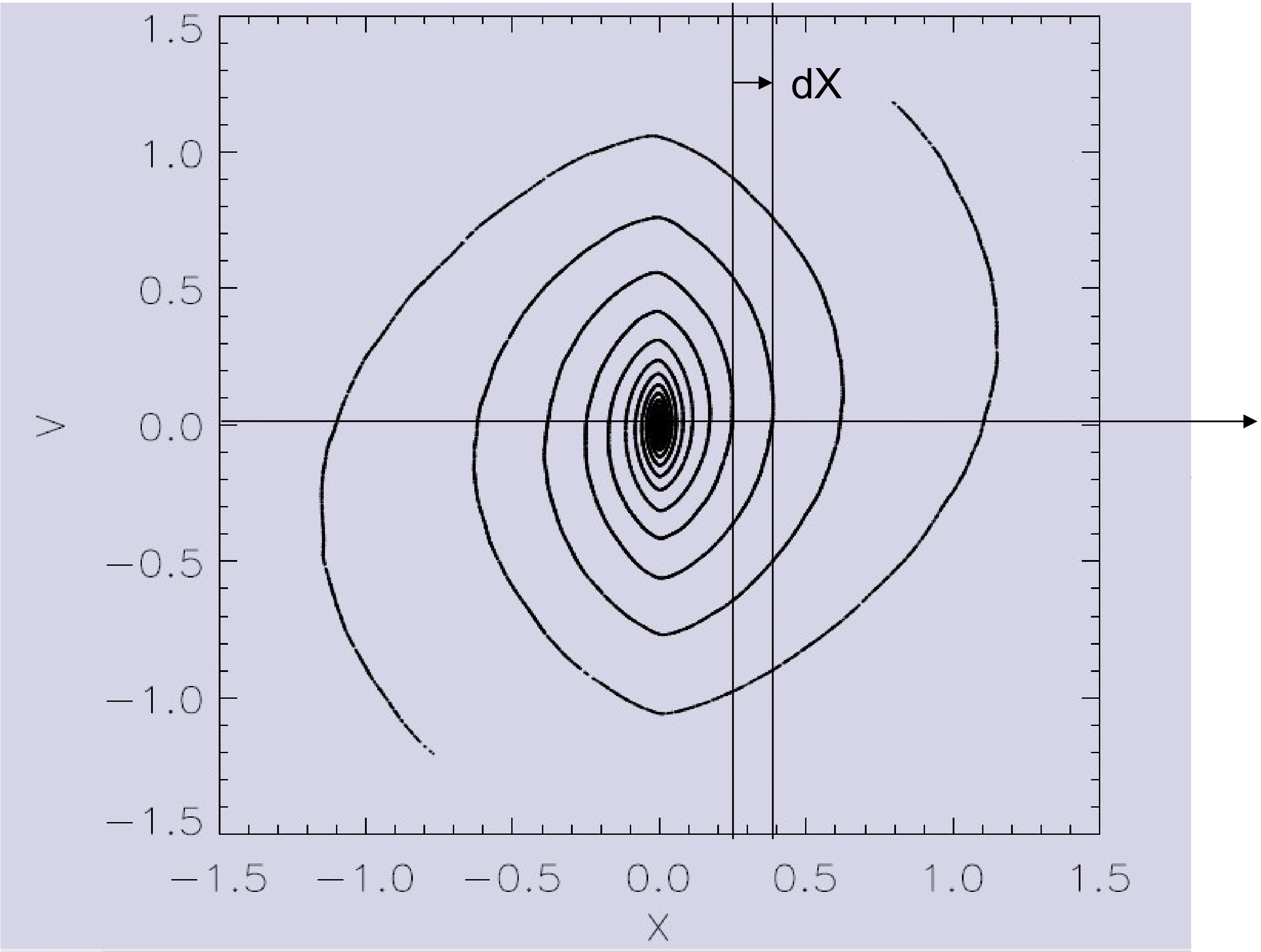
Requiring that all functions are power laws implies that:

$$\frac{dR}{R_0}\propto R_0^{\frac{1}{\alpha_2}}$$

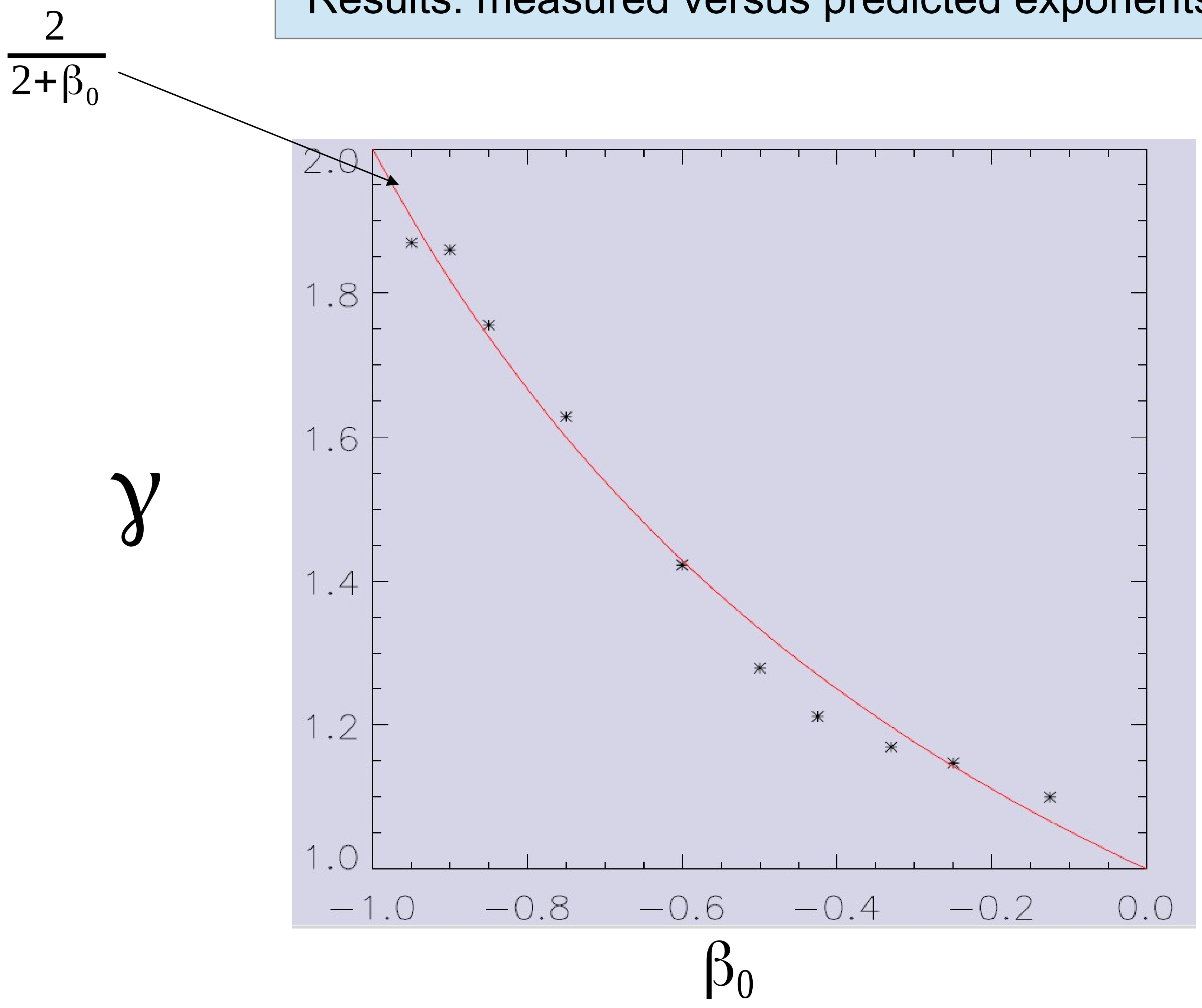
Series of numerical simulations with power law initial conditions

Initial density: $\rho_0 \propto x^{\beta_0}$

Fold spacing prediction: $dR \propto R_0^{1+\frac{1}{\alpha_2}} \propto R_0^{\frac{2}{\beta_0+2}}$ Numerical estimation: $dR \propto R_0^{\gamma}$



Results: measured versus predicted exponents



A general self similar conjecture for warm systems

For 3D spherically symmetric systems a general
Conjecture appears at small distance from the center

We will see that the self similar solution predicts the proper
Geometry for the folds even when the initial conditions are
Not dynamically cold.

This suggests that the self similar solutions of the Vlasov-Poisson
System are not important only for cold systems but are of interest
Also in the general case.

Symmetrically spherical system with Angular momentum

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial r}v_r+\left(\frac{J^2}{r^3}-\frac{\partial \Phi}{\partial r}\right)\frac{\partial f}{\partial v_r}=0$$

$$f(r,v_r,J^2,t)=g(r,v_r,t)\,\delta(J^2-J_0^2)$$

$$\frac{\partial g}{\partial t}+\frac{\partial g}{\partial r}v_r+\left(\frac{J_0^2}{r^3}-\frac{\partial \Phi}{\partial r}\right)\frac{\partial g}{\partial v_r}=0$$

The problem is very similar to the 1D problem

Two asymptotic regimes

$$r \ll 1 \;\rightarrow\; force \;\; dominated \;\; by \;\; \frac{J_0^2}{r^3} \;\; term \;\rightarrow\; \beta=-4$$

$$r \gg 1 \;\rightarrow\; force \;\; dominated \;\; by \;\; -\frac{\partial \Phi}{\partial r}$$

Re-scaling of potential of power law force field

$$(2+n\alpha_2)F+(1+\alpha_2)\frac{\partial F}{\partial \mathbf{x_2}}\mathbf{x_2}+\alpha_2\frac{\partial F}{\partial \mathbf{v_2}}-\frac{\partial F}{\partial \mathbf{x_2}}\mathbf{v_2}+k\,\mathbf{x_2}^{\beta}\frac{\partial F}{\partial \mathbf{v_2}}=0$$

$$\Phi=k\,x^{\beta+2}\quad,\quad x_2=\lambda_1 x_2\quad,\quad v_2=\lambda_2 v_2\quad,\quad k\,x_2^{\beta}=\frac{1}{2}\rightarrow\quad \lambda_1=\lambda_2=k^{-\frac{1}{\beta}}$$

$$-2<\beta<0\quad\begin{cases} k>0 & \textit{attraction}\\ k<0 & \textit{repulsion}\end{cases}$$

$$\beta<-2\quad\begin{cases} k>0 & \textit{repulsion}\\ k<0 & \textit{attraction}\end{cases}$$

Consequence of changing force sign on the solution contours

$$\lambda^{\beta} = -1$$

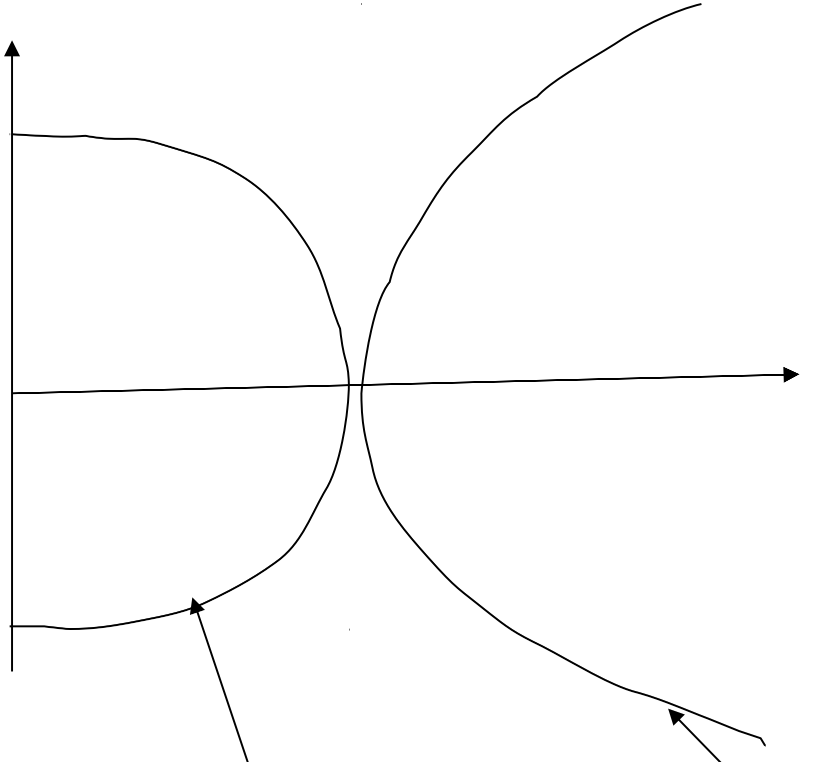
Changes of sign in the equation

$$R^{-\frac{\beta}{\beta+2}} \int |\cos\psi|^{-\frac{\beta}{\beta+2}} d\psi \approx R^{\frac{-\beta}{\beta+2}} \int d\psi \qquad Small \quad \psi \;\rightarrow\; d\psi \approx \tan\psi = \frac{v}{x^{\frac{\beta}{2}+1}} \;\rightarrow\; \lambda^{-\frac{\beta}{2}} \frac{v}{x^{\frac{\beta}{2}+1}}$$

$$R^{\frac{-\beta}{\beta+2}} \int d\psi = C_0 \;\rightarrow\; x^{\beta+2} + v^2 \propto \left(\frac{v}{x^{\frac{\beta}{2}+1}} \right)^{-\frac{2(\beta+2)}{\beta}} \;\rightarrow\; \lambda^2 \left(-\lambda^{\beta} x^{\beta+2} + v^2 \right) \propto \lambda^{-\frac{\beta}{2}\frac{2(\beta+2)}{\beta}} \left(\frac{v}{x^{\frac{\beta}{2}+1}} \right)^{-\frac{2(\beta+2)}{\beta}}$$

$$\lambda^{\beta} = -1 \;\rightarrow\; -x^{\beta+2} + v^2 = -\left(\frac{v}{x^{\frac{\beta}{2}+1}} \right)^{-\frac{2(\beta+2)}{\beta}}$$

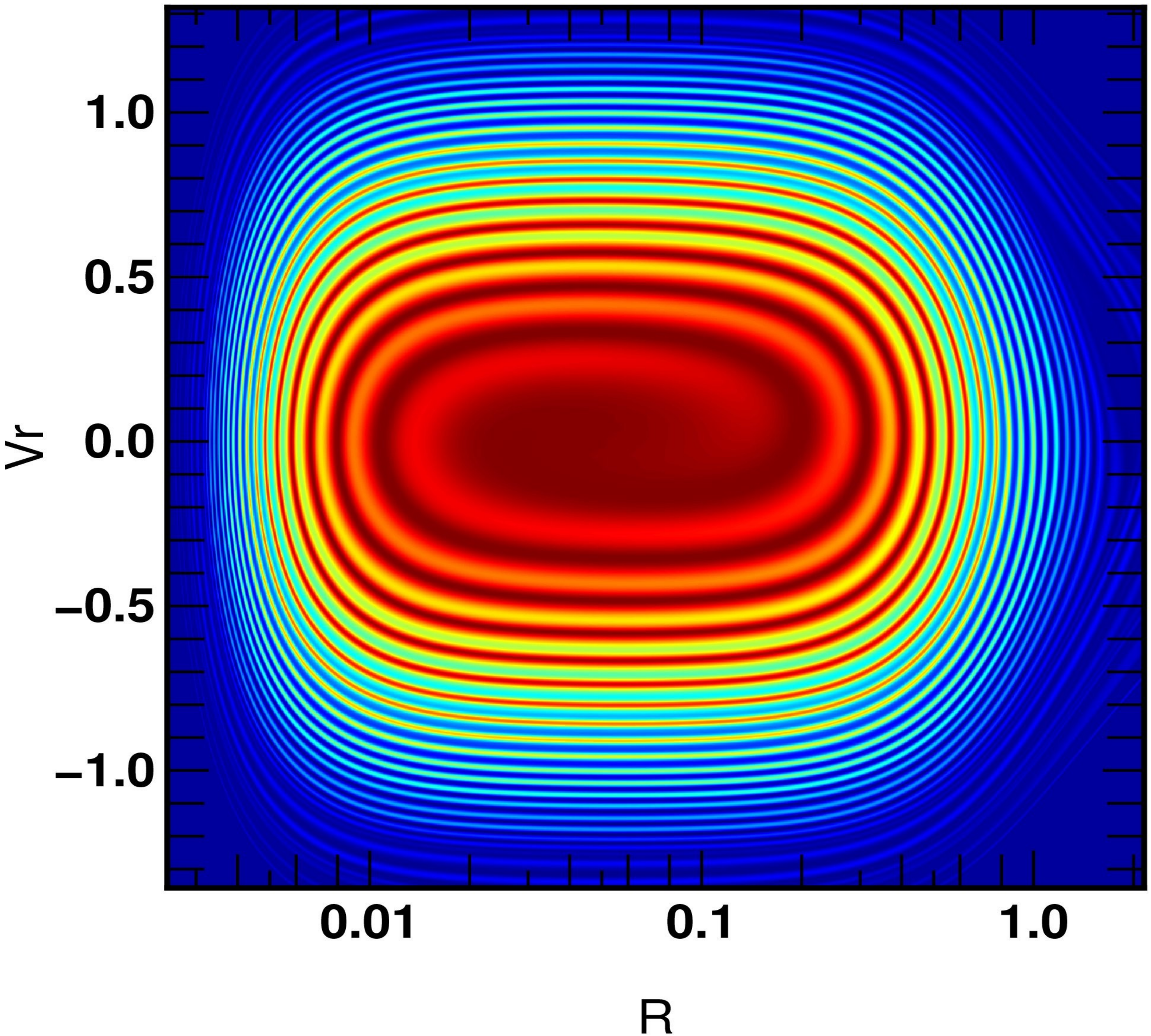
Illustration on the effect of force sign



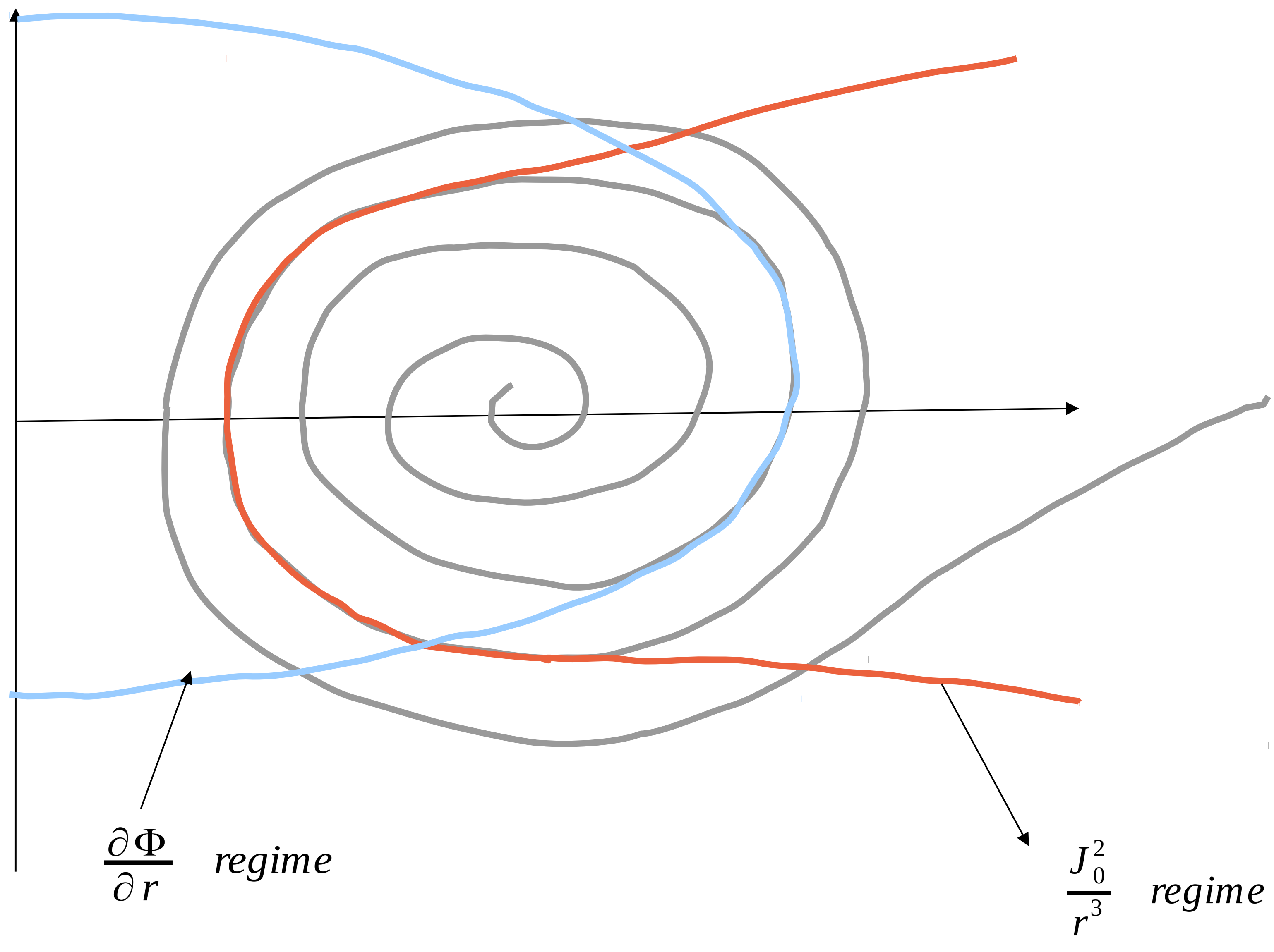
$\beta = -1$, $k > 0$ → $x + v^2 = 1$, $k < 0$ → $-x + v^2 = -1$

Numerical simulations: Thierry Sousbie, IAP

Spherical collapse with angular momentum

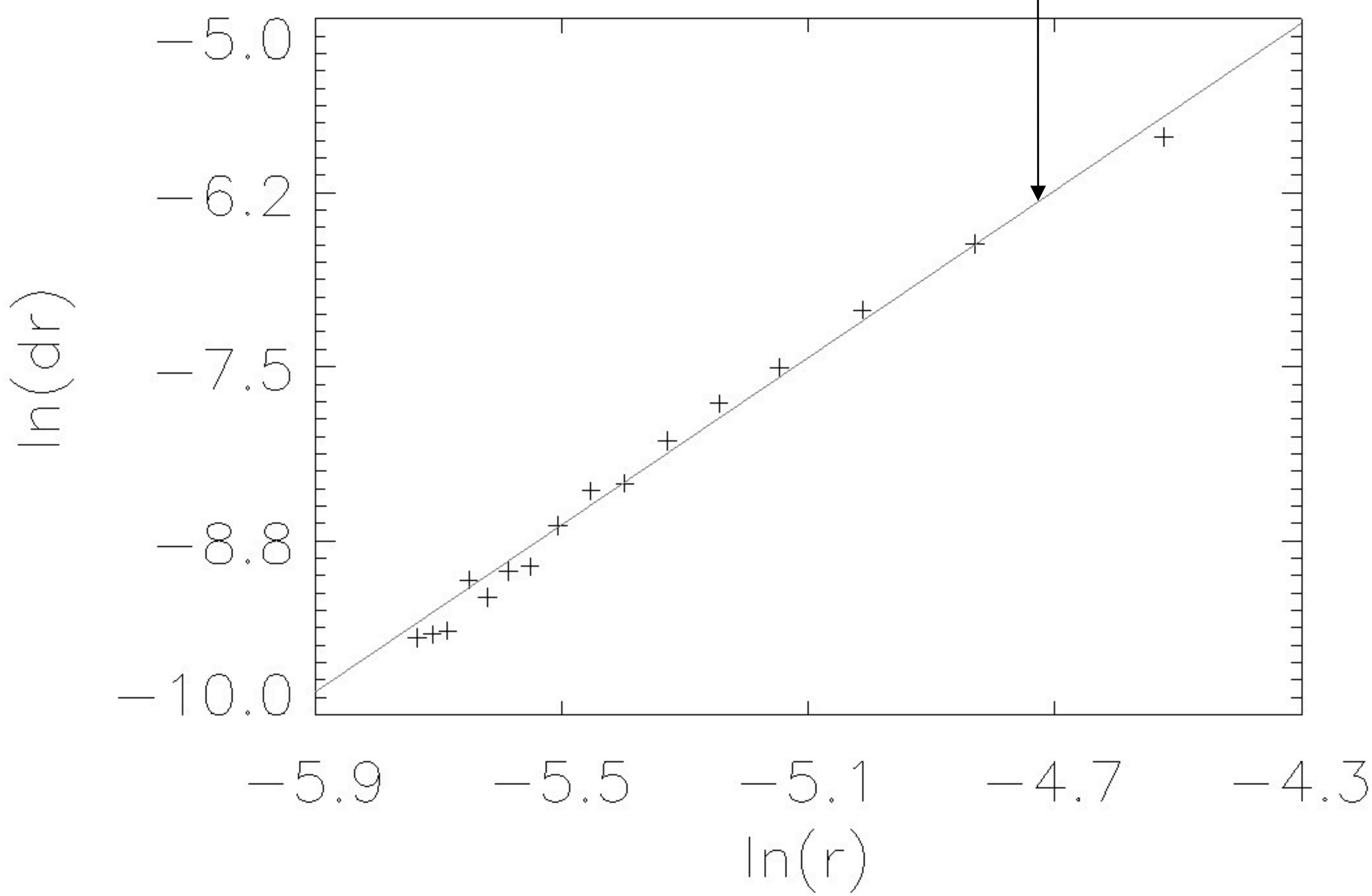


Folds with angular momentum



Small r force dominated by angular momentum repulsive term

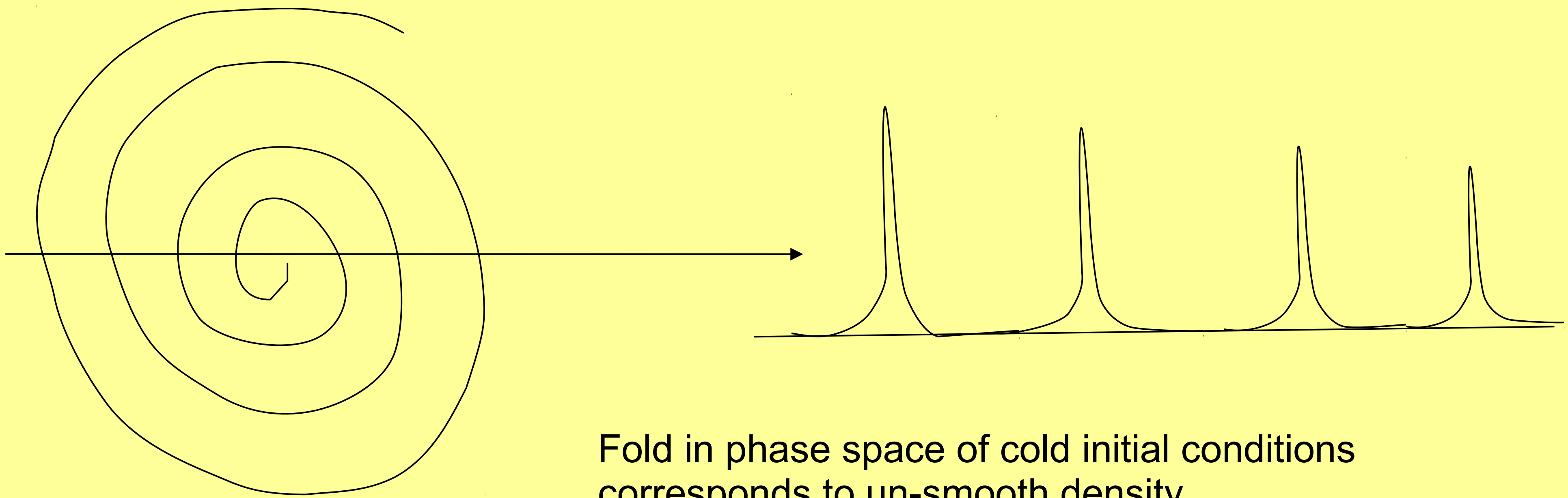
$$dr \propto r^{-\frac{\beta}{2}+1} \quad , \quad \beta = -4 \quad \rightarrow \quad dr \propto r^3$$



General statistical properties of solutions near equilibrium

Why do we observe some universal laws in numerical simulations ?

For smooth phase space density the expectation are all power laws



Fold in phase space of cold initial conditions corresponds to un-smooth density

Smoothing the phase space density breaks the auto similarity: no power law

Some auto similar properties are conserved in a smoothing process
Fold thickness is proportional to fold distance: a key property

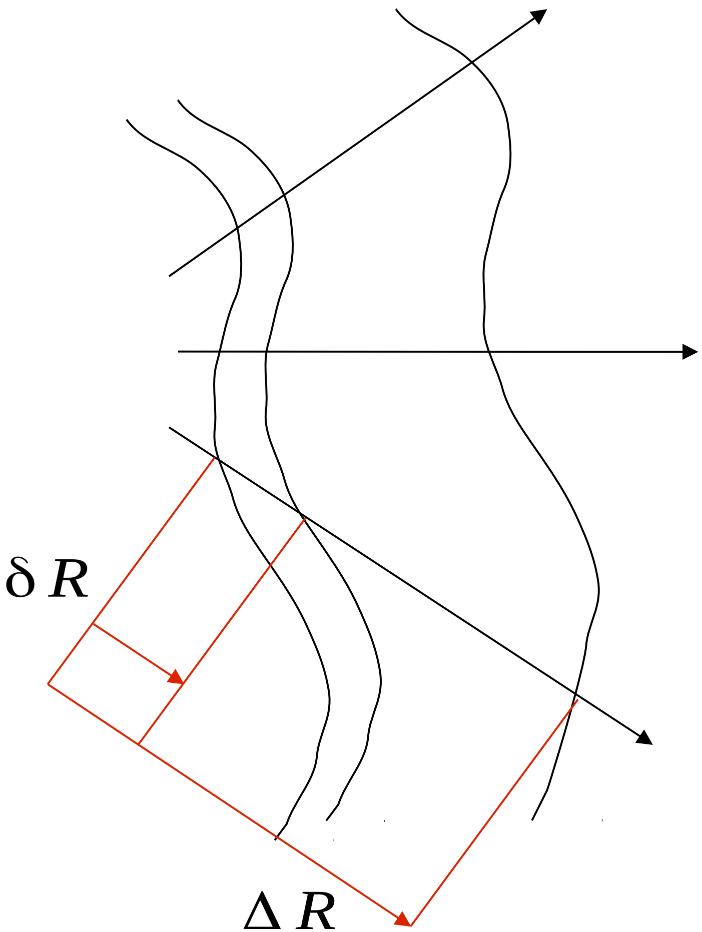
Generalized spiral $R = s(\psi_1, .., \psi_n)$

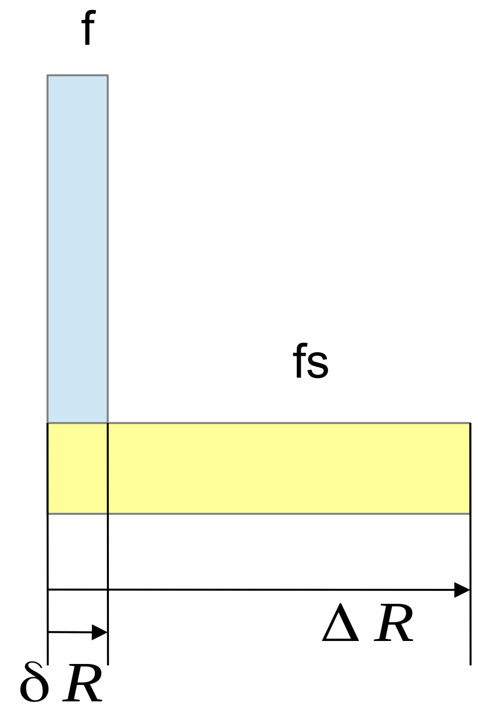
In a given dimension associated with ψ_k a fold corresponds to $\psi_k = \psi_k + 2\pi$

A variation of the initial condition $d\psi_k$ corresponding to the fold thickness $\psi_k = \psi_k + d\psi_k$

$\psi_k \gg 2\pi$ ⟶ Linearization $\delta(2\pi) \propto \delta(d\psi_k)$

Fold distance and fold thickness are proportional

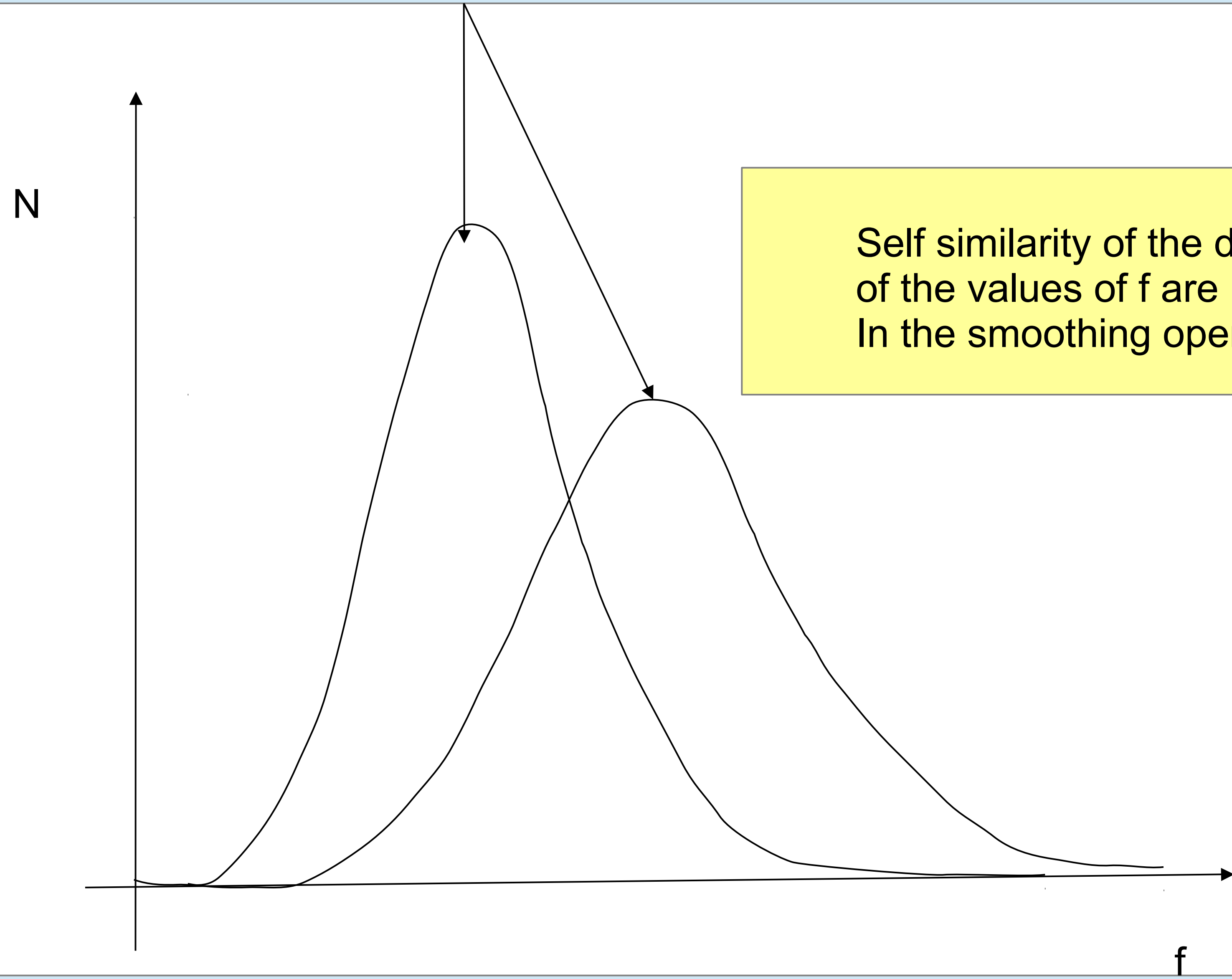




$$\frac{\delta R}{\Delta R} = Constant \longrightarrow$$

Constant scaling between
values and occurrence of f and fs

Histogram of f and fs are identical except for constant scaling factors



Self similarity of the distribution
of the values of f are preserved
In the smoothing operation

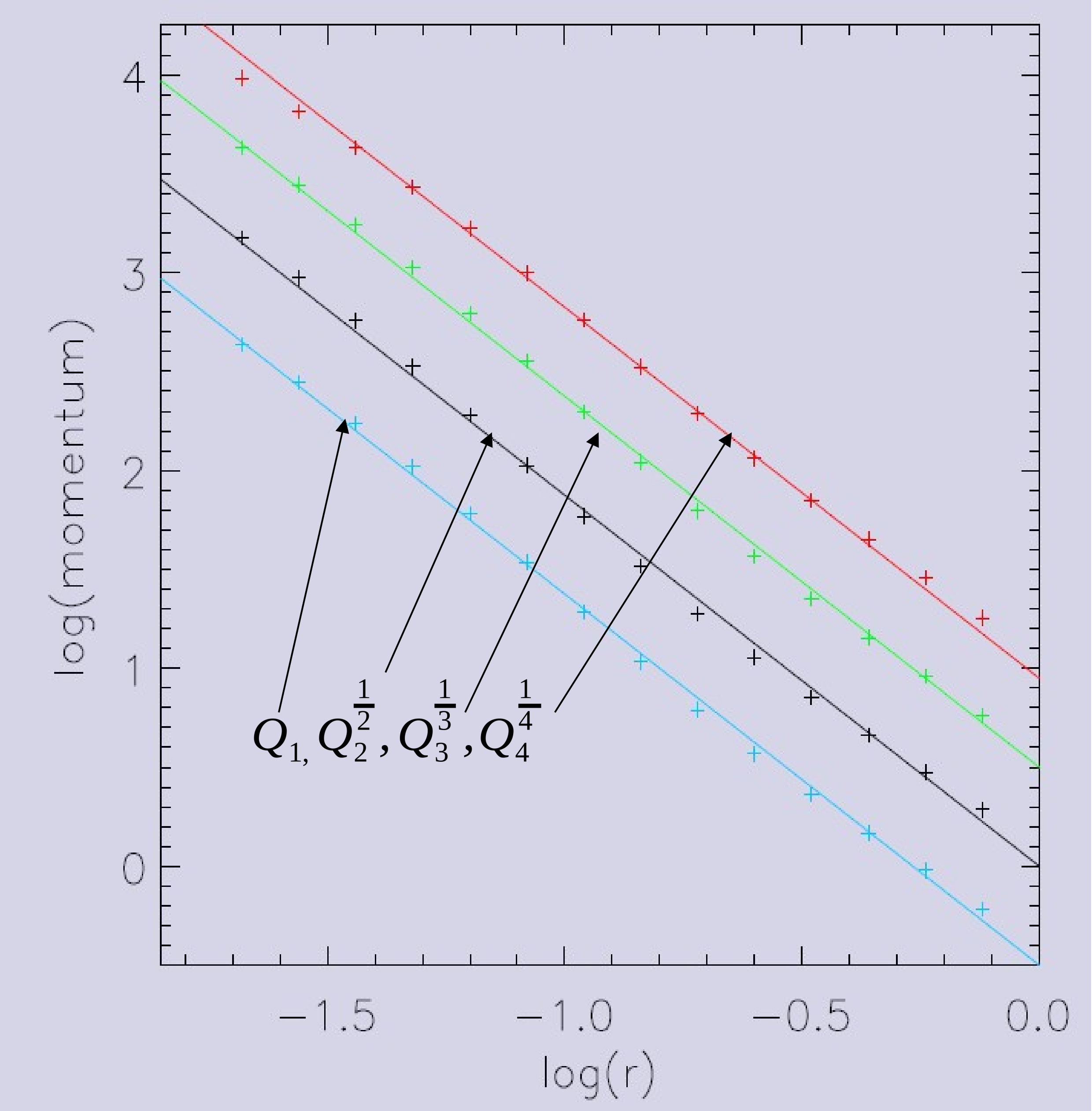
Consequence: the expectation of smoothed quantities derived from the probability distribution of f is a power law

Quantities derived from the smoothed probability distribution

$$Q_n = \int P(f) f^n df \approx \frac{k}{\sigma^3} \int f^n d^3 v$$

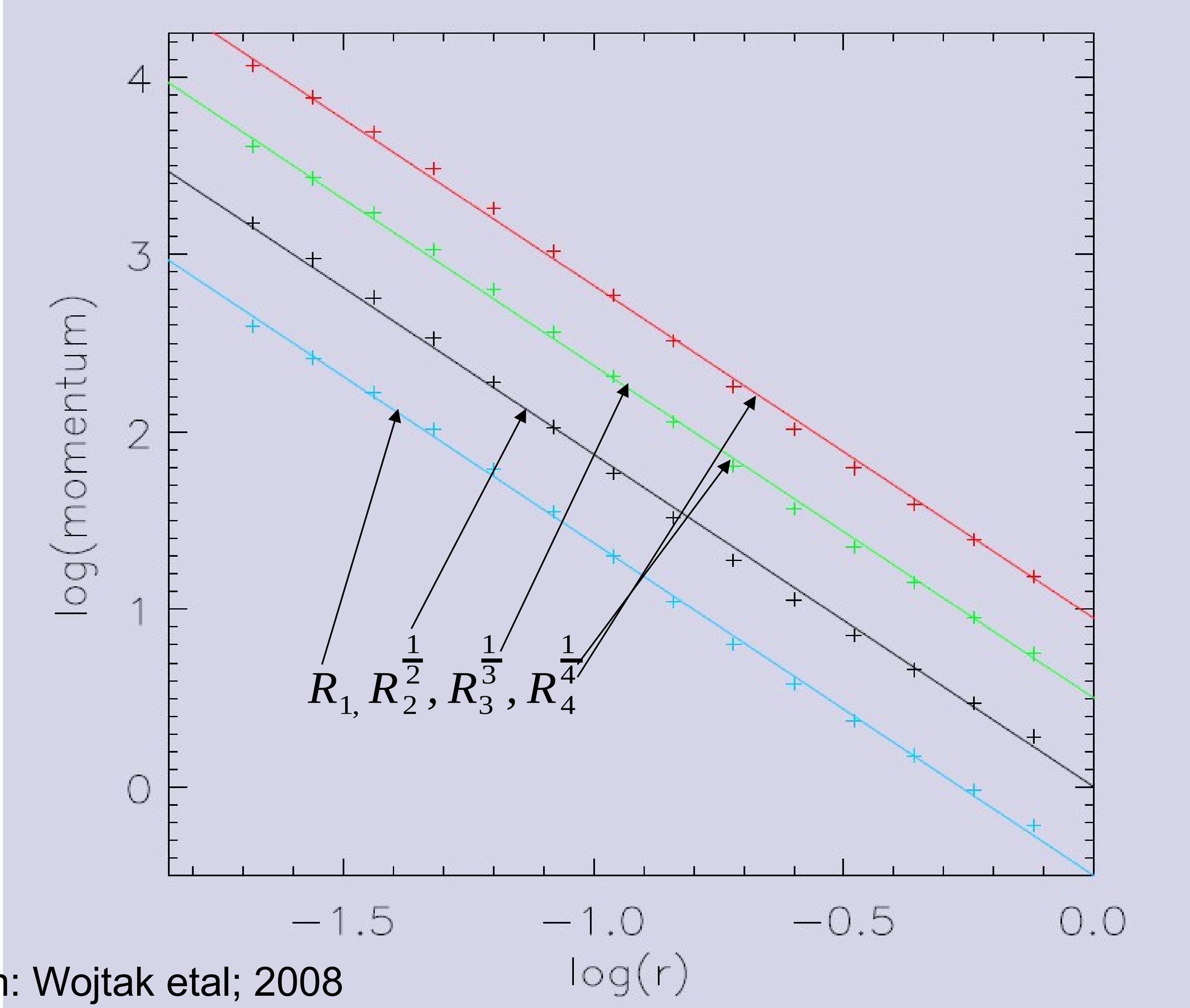
$$Q_n = t^{n\alpha_0} \left(\frac{r}{t^{\alpha_1}} \right)^{\gamma} \propto r^{-n\frac{15}{8}}$$

$$Q_1 \propto \sigma^{-3} \int_{-p\sigma}^{p\sigma} f\, d^3 v \approx \frac{\rho}{\sigma^3} \propto r^{-\frac{15}{8}}$$



Simulation data from: Wojtak etal; 2008

$$R_{nm} = \frac{Q_n}{Q_m} = \frac{\int f^n d^3 v}{\int f^m d^3 v} = r^{-(n-m)\frac{15}{8}}$$



Simulation data from: Wojtak etal; 2008

DM self similarity induced by baryons

At the begining of galaxy formation the initial DM halo is affected by the baryonic feedback, Supernovae winds, AGN,...resulting in a re-shaping of the halo

The cuspy halo center is destroyed and replaced with a core

In this starburst episode the momentum driven winds impose an equilibrium condition

The violent re-shaping of the halo and the equilibrium condition is consistent with a new self similar class

This new baryonic induced self similar model of DM halo explains and is consistent With a number of observational facts.

Equilibrium condition for the gas

(Murray, Qataert & Thompson, 2005)

Optically thick gas radiation momentum equal gravity:

$$\frac{G\,M\,M_G}{r_B^2} = \frac{L_M}{c}$$

Total momentum proportional to number of star and to gas mass

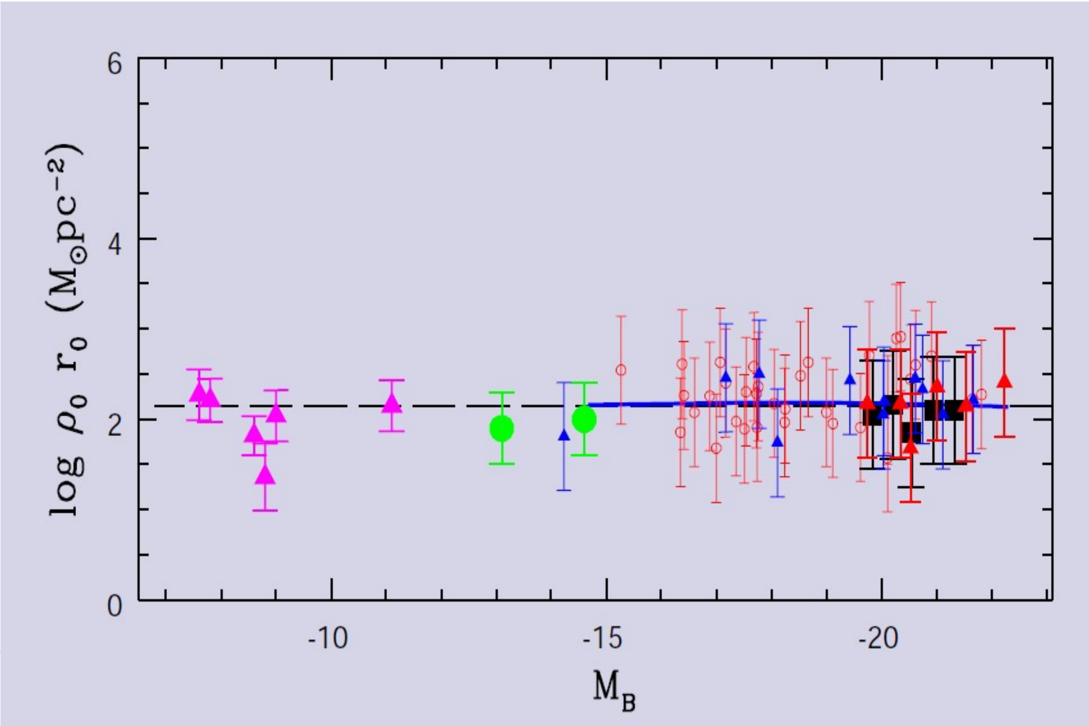
$$L_M \propto M_G \quad \rightarrow \quad \frac{G\,M}{r_B^2} = constant$$

Universal acceleration at a scale radius → new similarity class $\alpha_2 = 1$

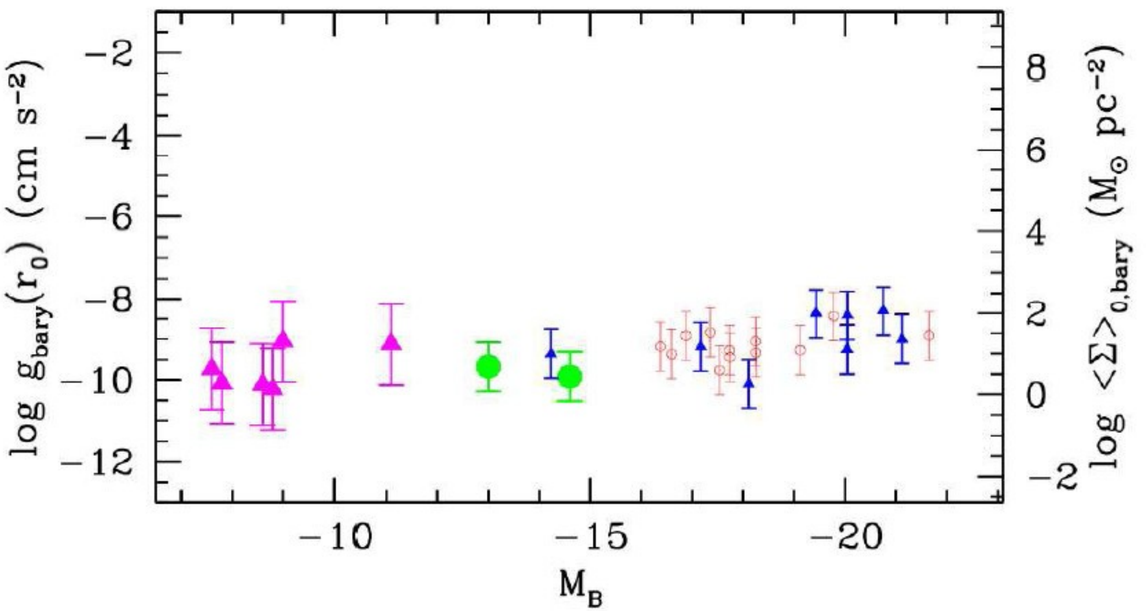
Constant DM acceleration → constant baryon acceleration

Former similarity class, infall on a seed mass Bertschinger, $\alpha_2 = -\frac{1}{9}$

Constant DM and baryons acceleration at a galactic scale radius



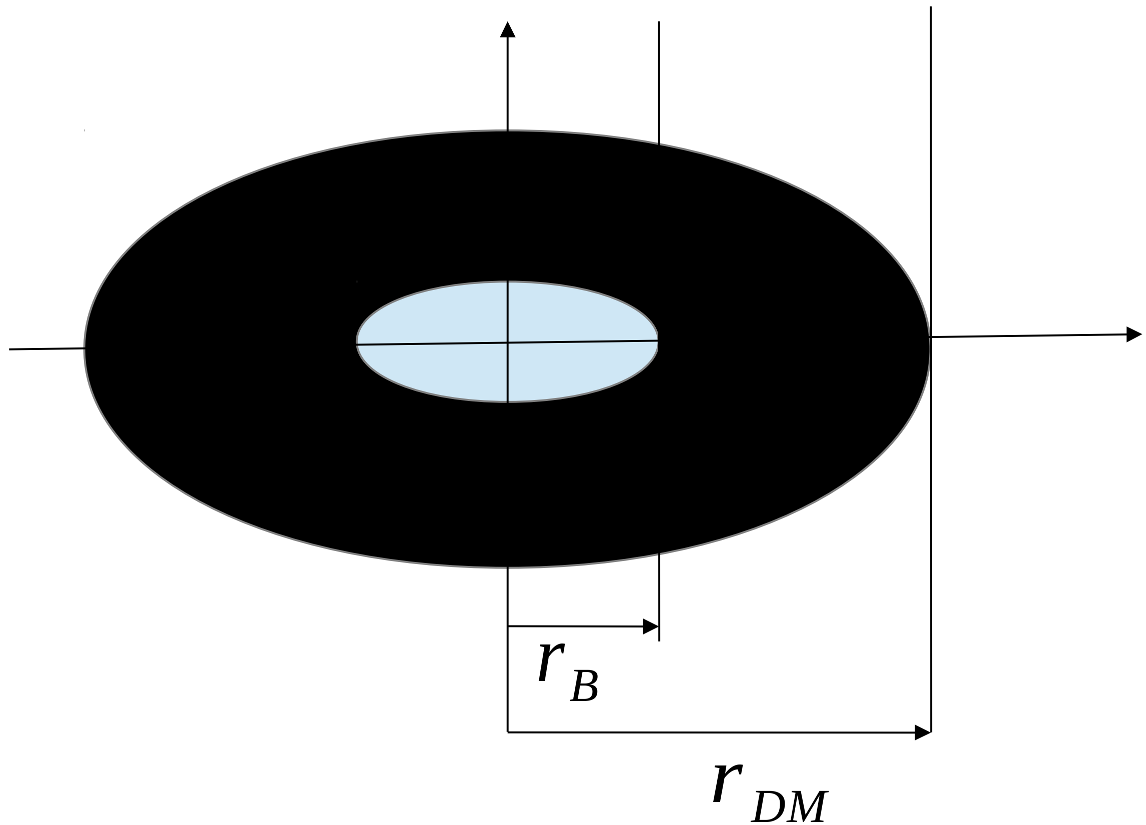
Acceleration at a galactic scale
For DM (Donato etal 2009)



Acceleration at a galactic scale
For baryons (Gentile etal 2009)

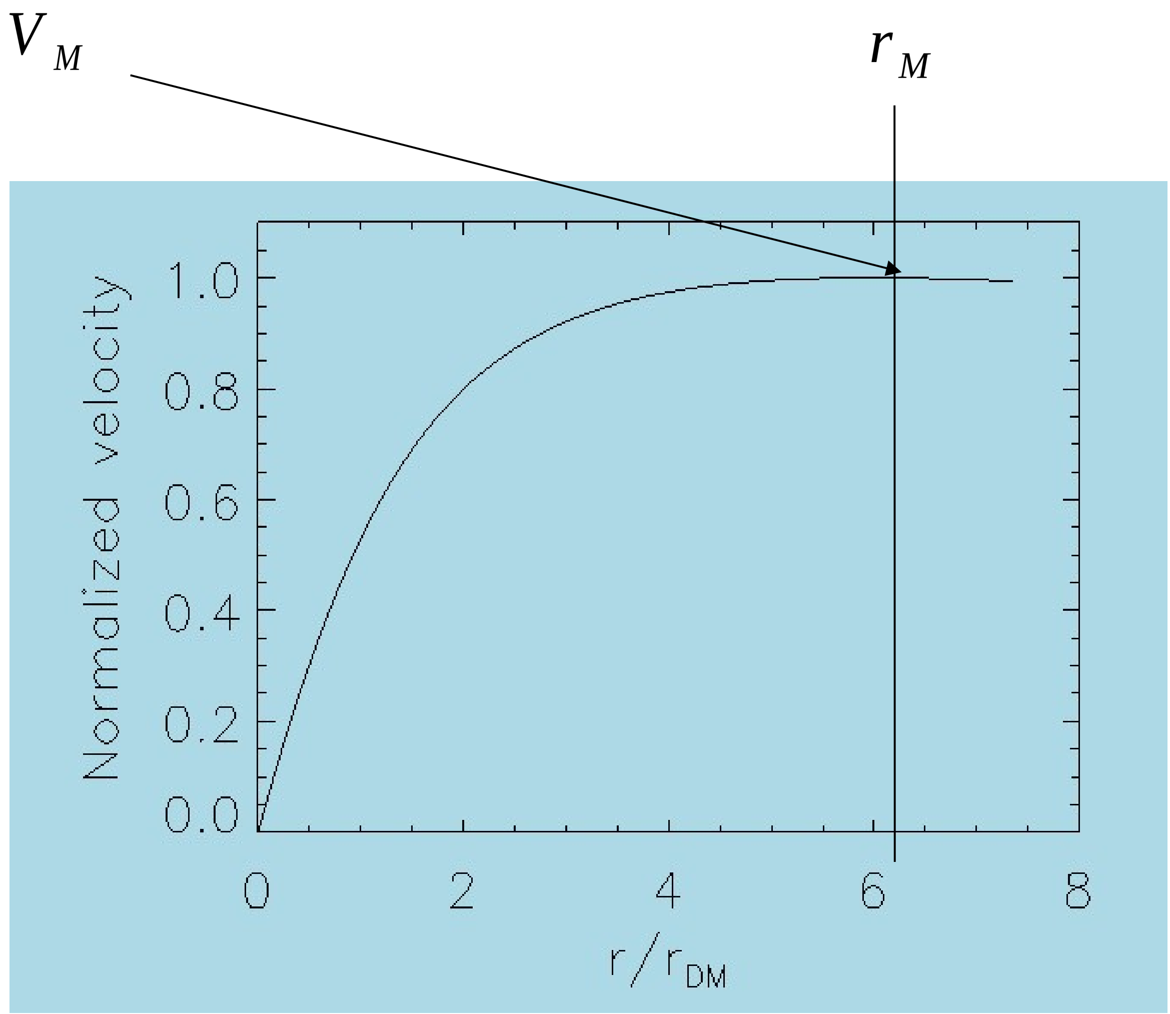
The scaling of DM and baryons distribution must be similar

$$r_B \propto r_{DM}$$



Donato etal. 2004 demonstrate that it is the case for a sample of about 40 galaxies

Velocity rotation curve



Maximum of the velocity curve situated at large distances

Self similarity →

$$\left\{ \begin{array}{l} \frac{V_M^2}{r_M} = c_0 \\ \\ \frac{G\,M_B}{r_B^2} = a_0^B \quad \& \quad r_B \propto r_{DM} \propto r_M \end{array} \right.$$

↓

$V_M^4 \propto G\,M_B$ Deep MOND regime

Transition between Newtonian & “Mond” regime at fixed acceleration also

But the gas equilibrium condition does not apply to clusters: unlike MOND the self similar model model that the similarity relation for velocities break beyond the scale of galaxies.

Conclusion

The analytic self similar solutions of the Vlasov equation are generalized folds in phase space.

The predicted structure of these folds is observed in numerical simulations

The statistical properties of the solutions are predictable and also observed In numerical simulations.

The baryons feedback introduce a specific similarity class, the properties of The associated solutions explains and connect a number of facts:

1) universal accelerations

2) proportionality between DM and baryons scale length

3) MOND like properties and the break of these properties at the scale of Clusters of galaxies.