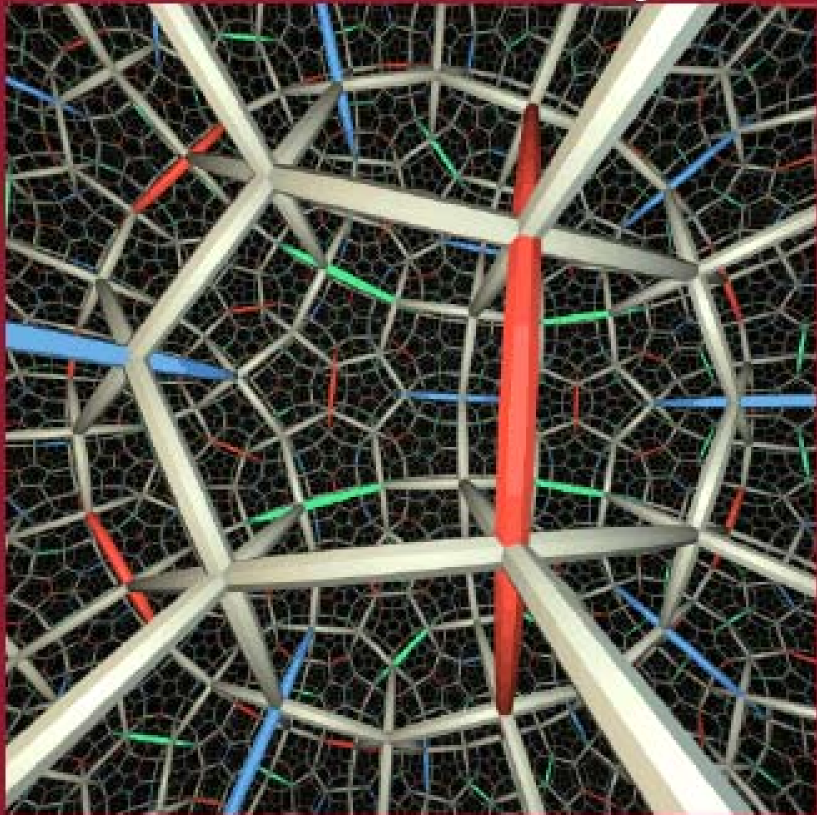
CMB modes and cosmic topology

Marc Lachièze-Rey



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Cosmic topology

"Only two things are infinite, the universe and human stupidity, and I'm not sure about the former."

Albert Einstein.

Cosmological questions :

- Is the universe finite or infinite ?
  - in space ?
  - in time ?.
- Is the universe homogeneous ?
  - isotropic ?

• • •

Cosmic topology

- **standard answers** OK with [almost] all observations

- **Cosmological Principe** :

"the Universe is homogeneous and isotropic" (at large scale)

- Time is finite in past (big bang or Planck era)
- Space is finite or infinite (← the values of Ω and Λ)

- The standard models assume that the global topology of space is simply connected.
- But general relativity tells nothing about topology.

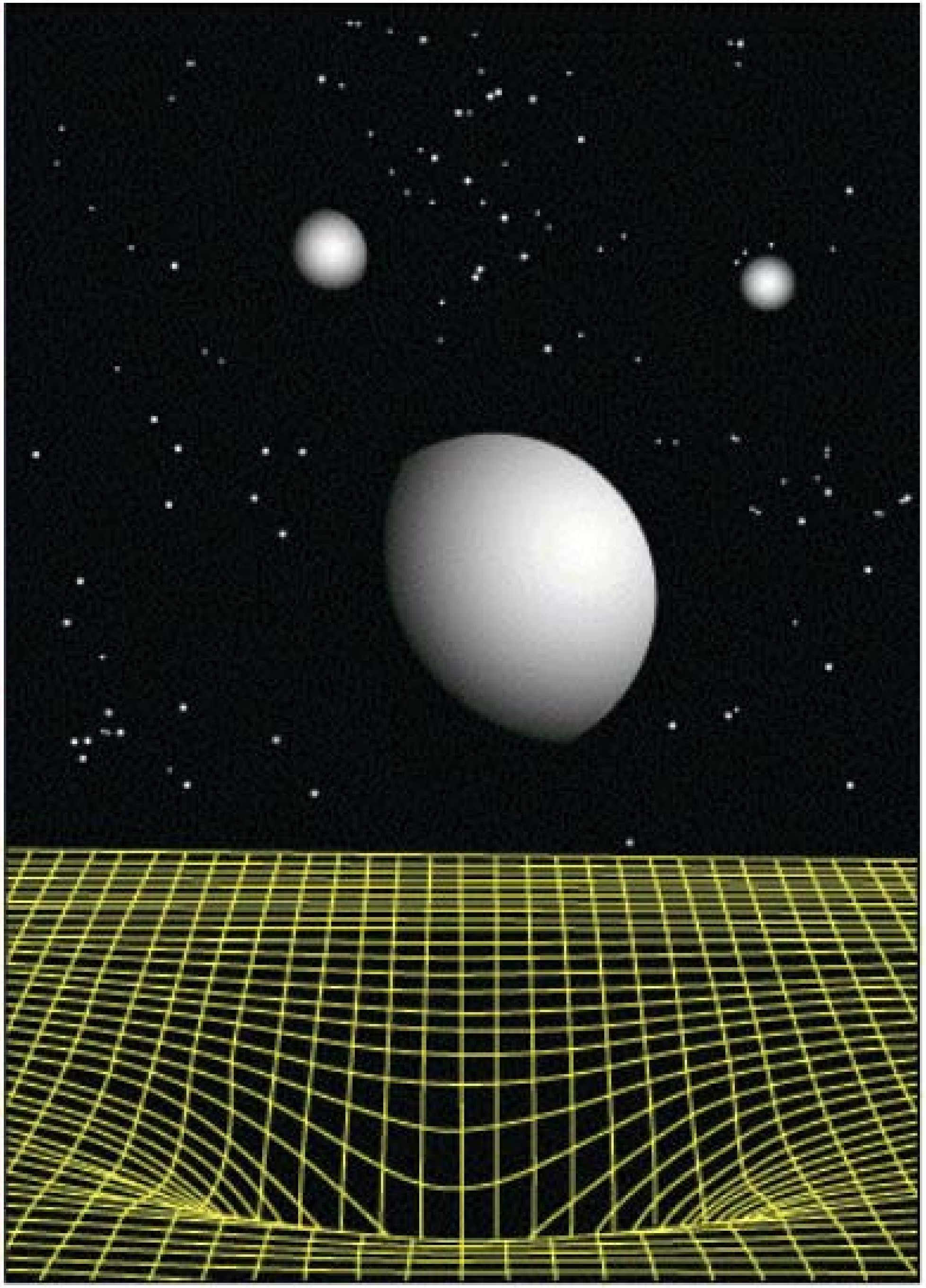
Main idea

General Relativity does not specify global properties of spacetime:

Einstein equations determine the (local) space-time curvature but Its (global) topology remains undetermined.

→ possibility of multi connected cosmological models (= MCMs)

Lachi



Multi connected cosmological models

standard cosmology:

space-time is topologically = $\Sigma \times R_{time}$

Spatial sections have the same topology of Σ :

MCM models assume multi connected spatial sections

- Instead of singly connected $\Sigma = R^3, S^3, H^3$

Simplest MCM : space = 3-cylinder and 3-torus

Start from R^3

G = group of translations of vector A
R^3/G = cylinder = R^3 modulo translation of vector A
Periodic in “horizontal” direction, with period |A|.

--> spacetime = $(R^3/\Gamma) \times R_{time}$

- 3 orthogonal directions of translations → 3-torus

Two questions

1 (mathematical) : What are the possible MCMs ?

←→

what are the multi connected 3d-Riemanian manifolds ?

- with constant curvature ?
- Required to be spatially orientable

2 (astrophysical) : What are the observable characteristics of the MCMs ?

Most appear similar to the standard (singly connected) models

→ difficult to discriminate.

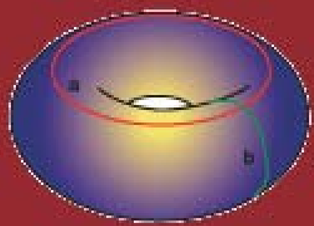
PLAN

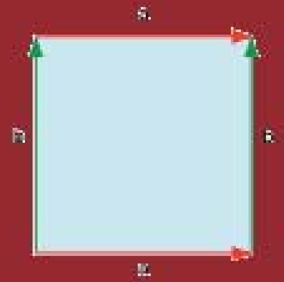
- Introduction & main ideas
- Multi connected SPACES
- Multi connected cosmology models
- Observational tests
- Calculation of modes

I - multi connected spaces

Pedagogical example : the 2-dimensional, flat torus

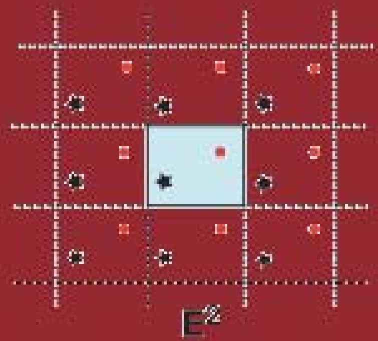
The manifold embedded in R3





The fundamental polygon

The universal covering surface R^2

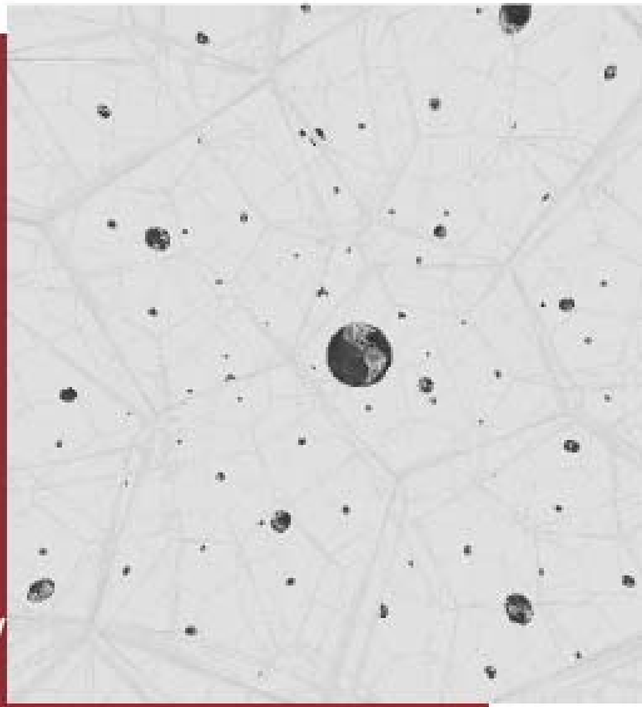


Moebius band

Non compact

Non orientable

Basic topological notions



universal covering surface (--> space) M*:

- homogeneous, with same (constant) curvature
- singly connected
- R^2 for the torus ; R^3, S^3 or H^3 in 3 dimensions)

helpful : appears as "observational space" in cosmology

fundamental polygon (--> polyhedron in 3 dimensions)

Tessellation of the UC by copies of the fundamental polygon (--> polyhedron)

holonomy group G:

the copies are transformed into each other by the holonomy group = subgroup of ISO(M*) acting freely and discontinuously.

- M=M*/G

Classification of MC spaces

MC spaces may be compact (torus) or not (cylinder)

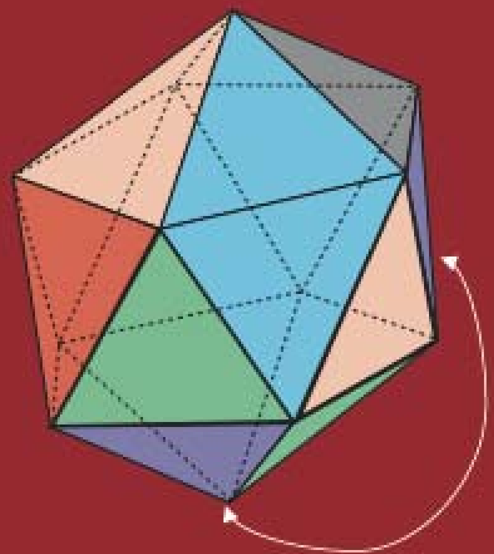
Spatial curvature (from cosmological observations) may be

- Flat (=Euclidean) : uc $=R^3$
- Spherical (spherical spaces = sherical forms) : uc $=S^3$
- Hyperbolic : uc $= H^3$

Classification not achieved (powerful theorems exist)

Classification of 3D Space forms

- Universal covering space M → E^3, S^3, H^3
- Fundamental Polyhedron FP
- Holonomy group G



$g \in G$ → $M_G = M/G$

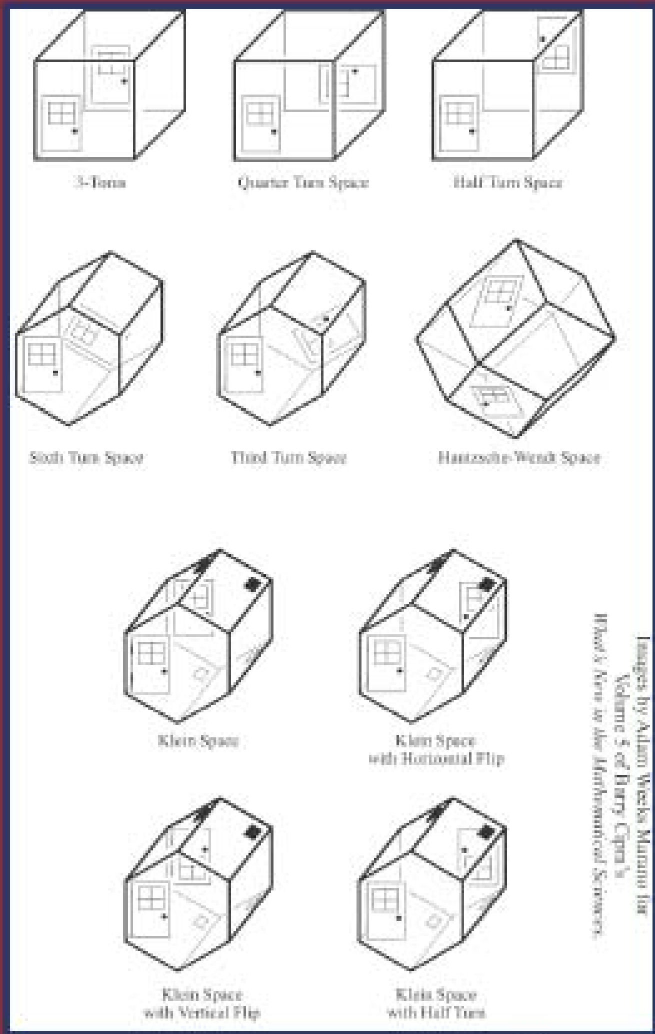
Flat (Euclidean) space forms

UC = R^3

- G = ISO(3) = E^3 x SO(3): combinations of translations, glide reflections, screw motions

Seven open (infinite volume).

Ten closed (finite volume).
6 orientable among them



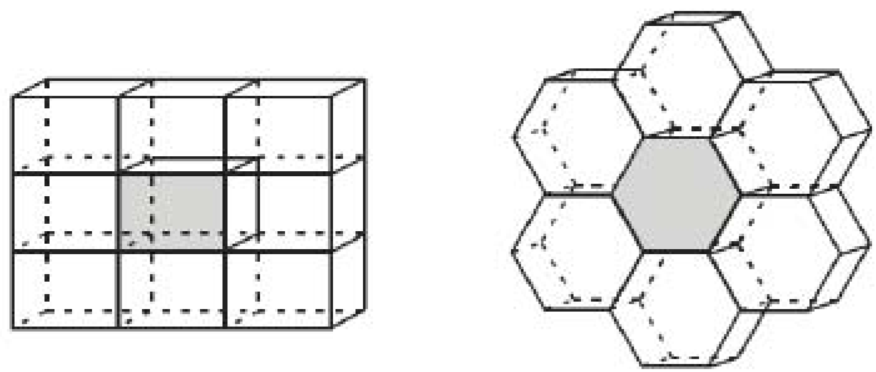


Figure 8. Tesselation of $\mathbf{R}^3$ by parallelepipeds or hexagonal cells

Thurston, Weeks (1970 -)
for CHM

Hyperbolic spaces

- universal covering space = H^3 (infinite in every direction)
- not completely classified (some closed, some open)
- Theorem : the volume of space (in curvature radius units) is fixed by the topology →

  classify them by increasing volumes
- The space with lowest volume has recently be found

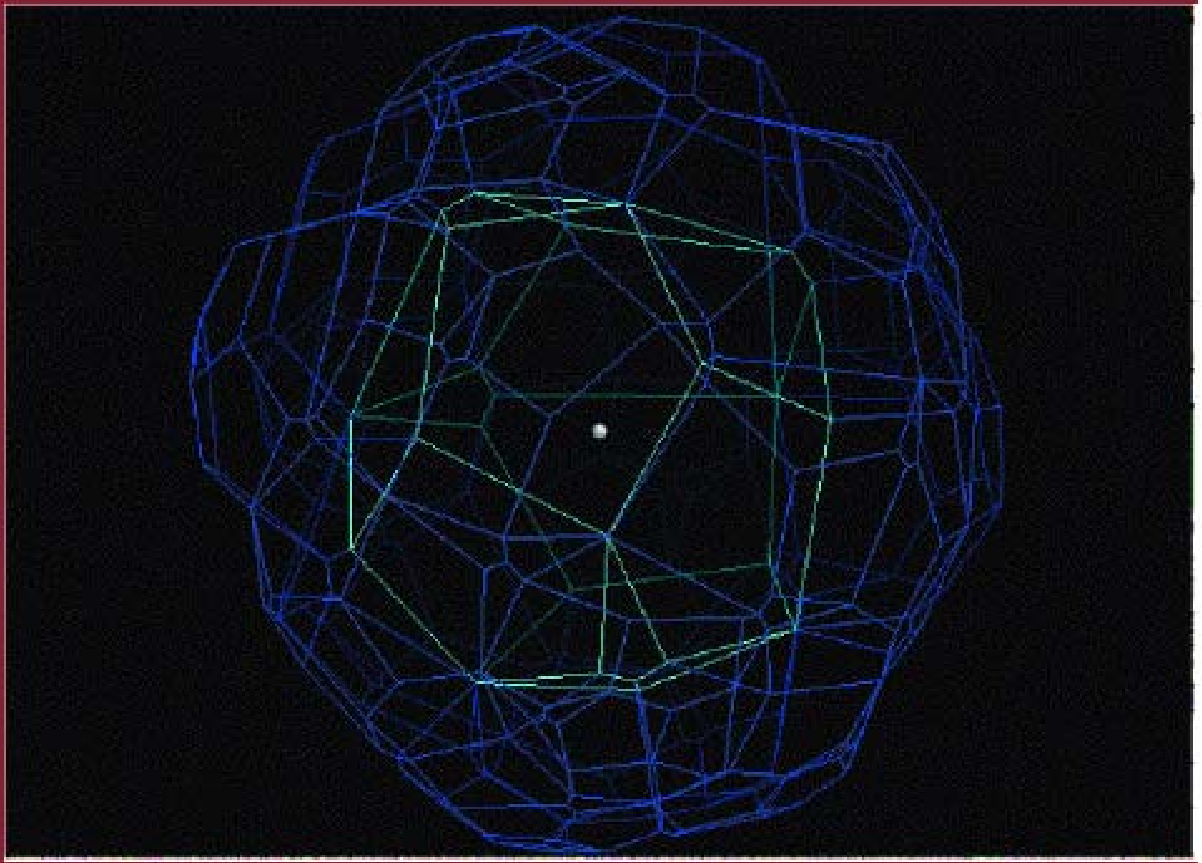
Weeks Space : V=0.94272

the smallest hyperbolic space !

- FP has

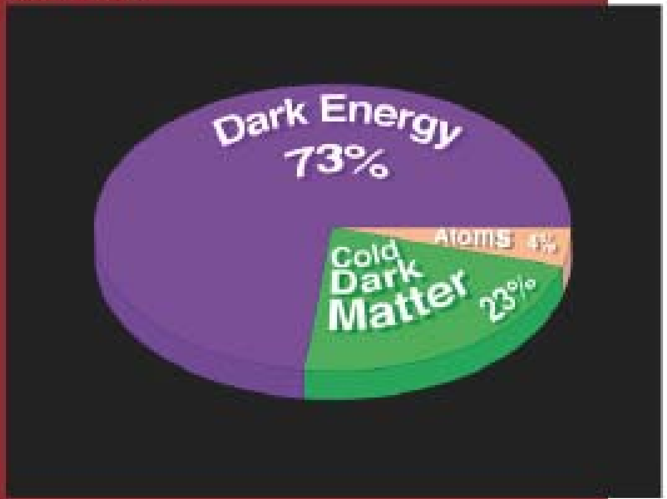
26 vertices
18 faces
(12 pentagons ,
6 quadrilaterals).

- Recently proved to be the smallest !



Spherical spaces (=spherical forms)

Natural unit = radius of the embedding space (uc) S^3
= [spatial] curvature radius



constrained by observations

(CMB, large scale structures…)

→ « nearly flat »

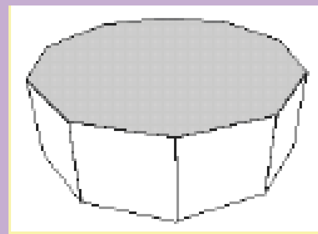
→Only some spherical forms possible !

→Most favoured : PDS

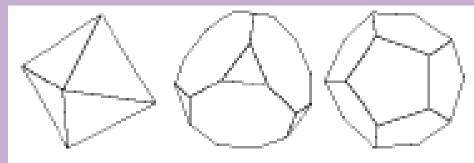
Spherical spaces

- universal covering space = S^3
- Lens Spaces S^3/Z_p

- Prism Spaces S^3/D_m



- Polyhedral Spaces S^3/T, S^3/O, S^3/I



Polyhedral spaces

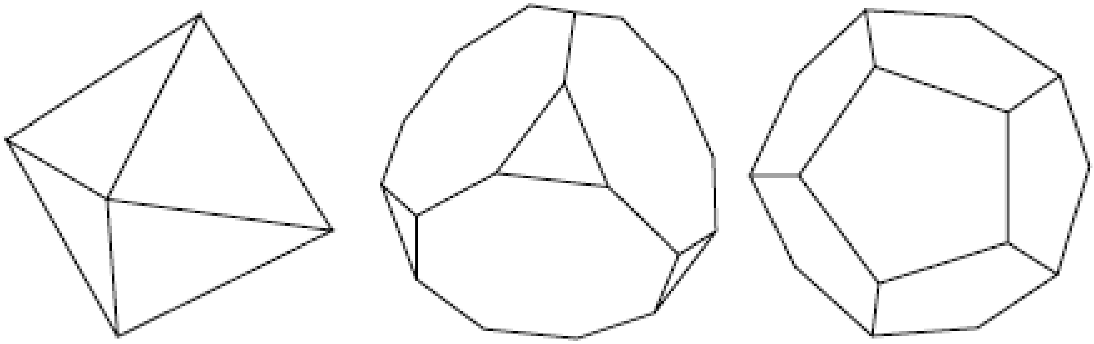
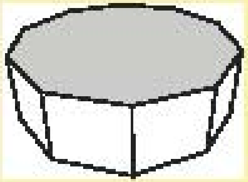
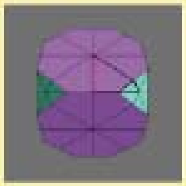


Figure 4. Fundamental domains for three single action 3–manifolds. From left to right, the regular octahedron, the truncated cube and the regular dodecahedron which respectively correspond to the spaces generated by the binary tetrahedral group T^*, the binary octahedral group O^* and the binary icosahedral group I^*.

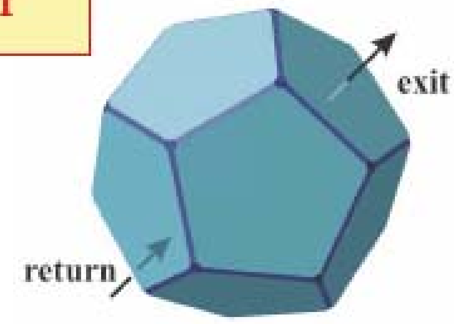
Nearly Flat Spherical WP Spaces

	Lens	Prism	Tetrahedral	Octahedral	Dodecahedral
FP					
Holonomy group	Z_p	D^*_m	T^*	O^*	I^*
$\frac{Volume}{Volume(S^3)}$	1/p	1/4m	1/24	1/48	1/120
Ω_{min} for observability	$1+1/n^2$	$1+1/4m^2$	1.025	1.015	1.009

Recall : $\Omega_{tot} = 1.054^{+0.048}_{-0.041}$ => k =+1

Poincaré Dodecahedral space S^3/I^*

FP = 12 faces regular (spherical) dodecahedron



identify the opposite pentagonal faces ,
after rotating by 1/10th turn in the clockwise direction around the axis orthogonal to the face.

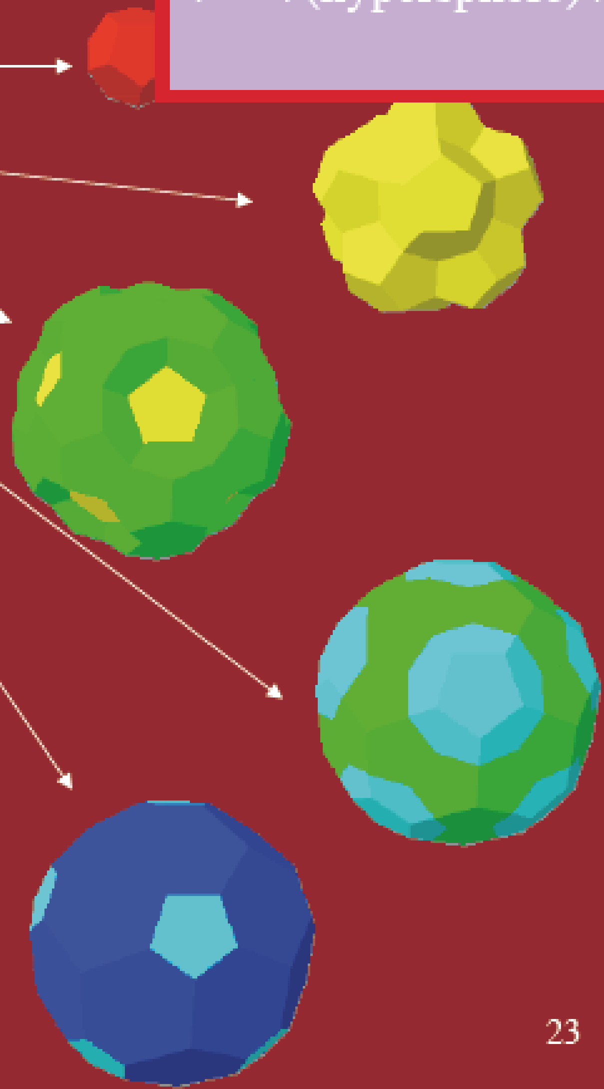
120 successive operations !

120 copies

tile S^3

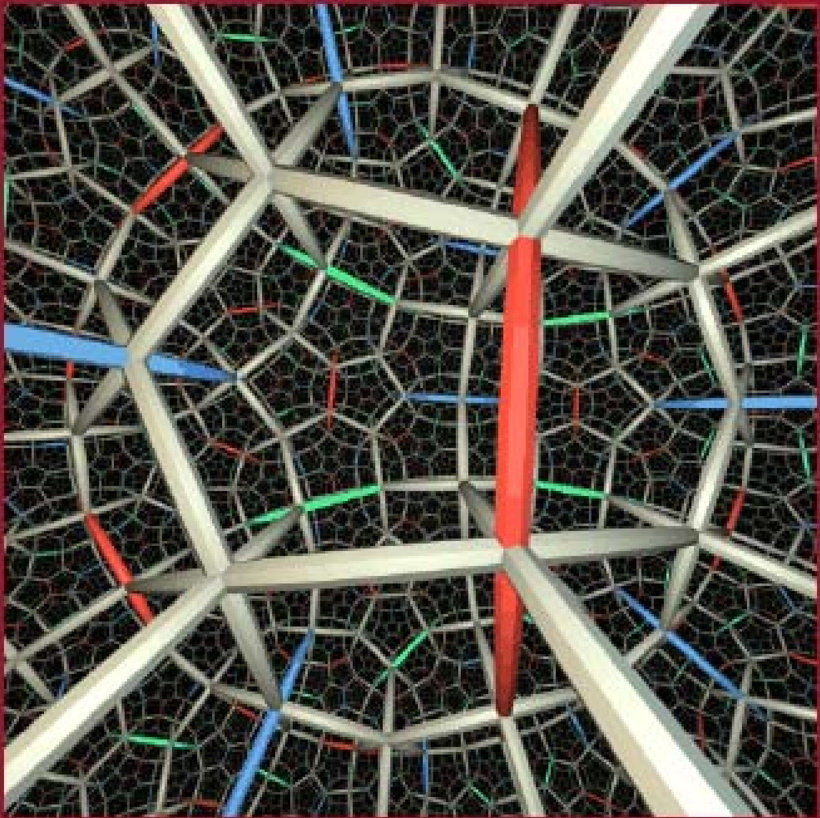
$V = V(\text{hypersphere}) / 120$

Layer	*Distance from O*	*Number of dodec.*
1	0	1
2	$\pi/5$	12
3	$\pi/3$	20
4	$2\pi/5$	12
5	$\pi/2$	30
6	$3\pi/5$	12
7	$2\pi/3$	20
8	$4\pi/5$	12
9	π	1
Sum		**120**



Lachièze-Rey, 2008

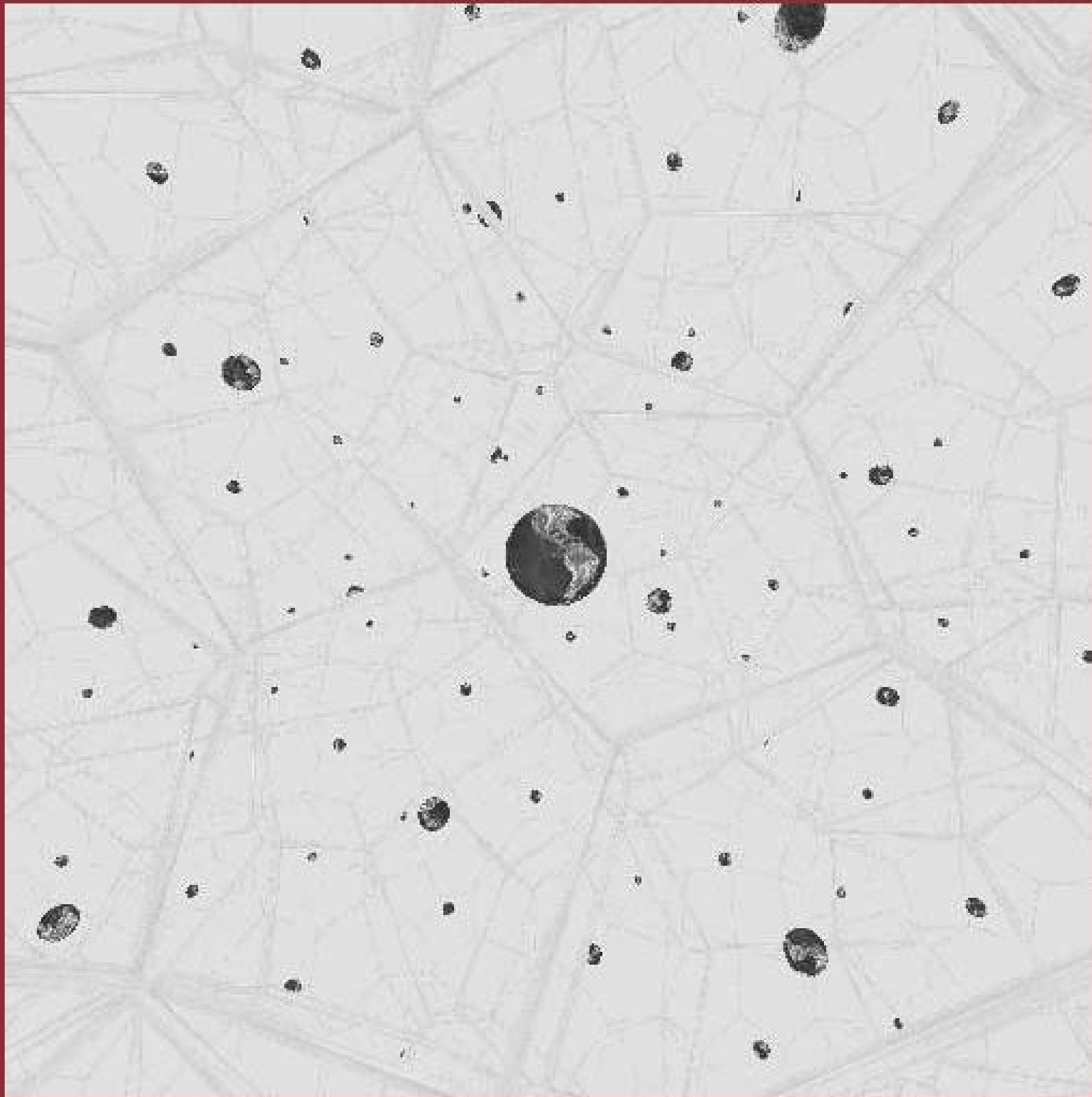
Poincaré dodecahedric space = PDS



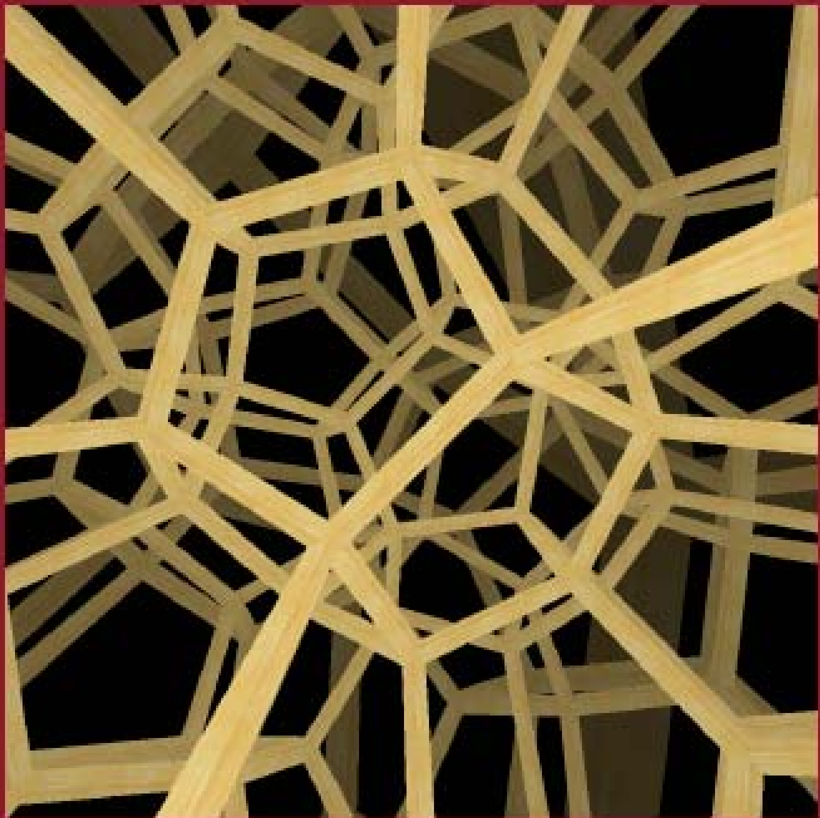
120 copies

tile S^3

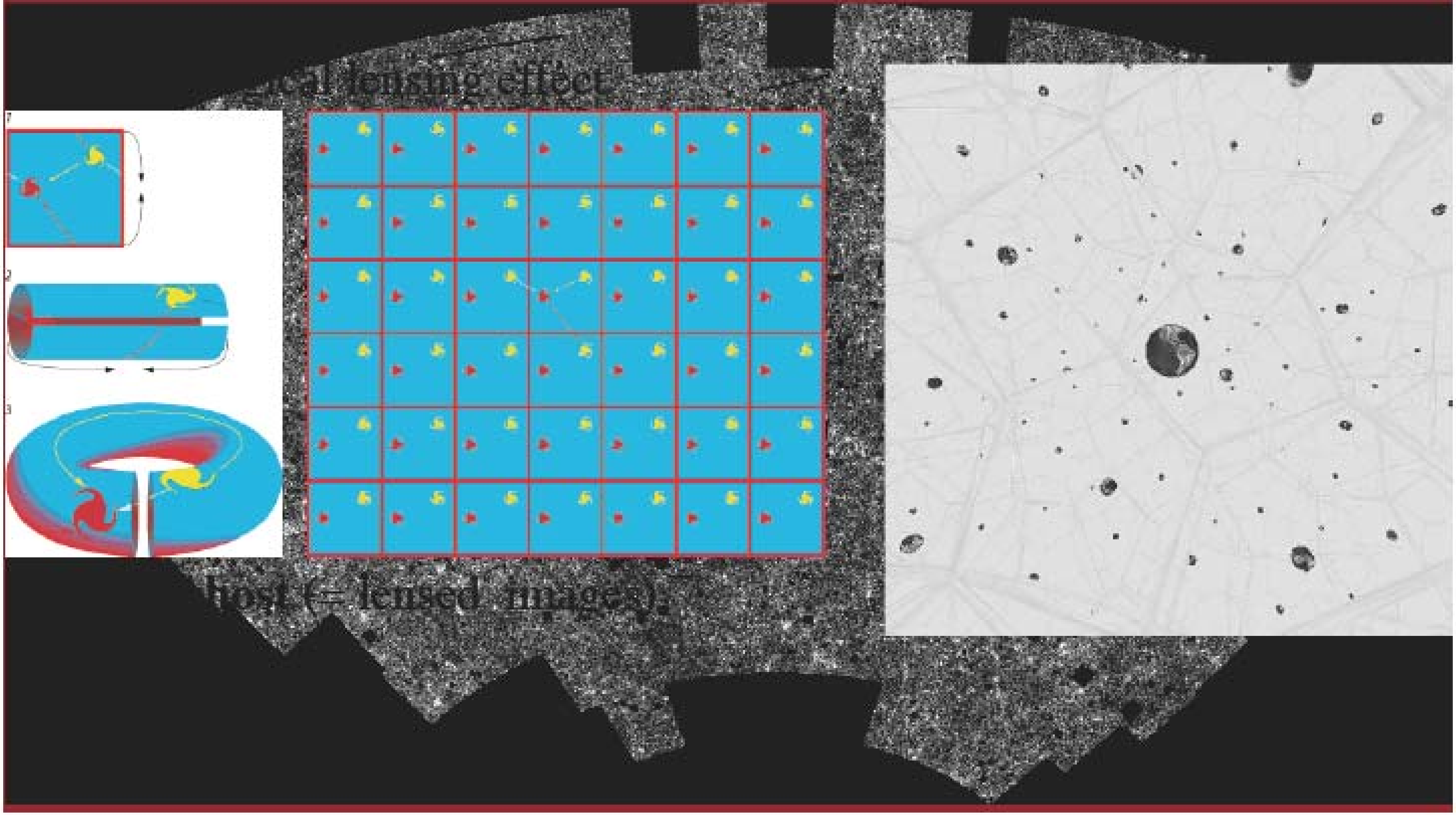
$V = V(\text{hypersphere}) / 120$



Poincaré Dodecahedral space S^3/I^*



Test Cosmic Topology with large scale structures



Cosmic crystallography : [or correlated pairs , in hyperbolic case]

→L > 650 h^{-1} Mpc (for flatcase)

Present catalogs not wide enough for definitve conclusions

→ wait for next surveys

Test with CMB

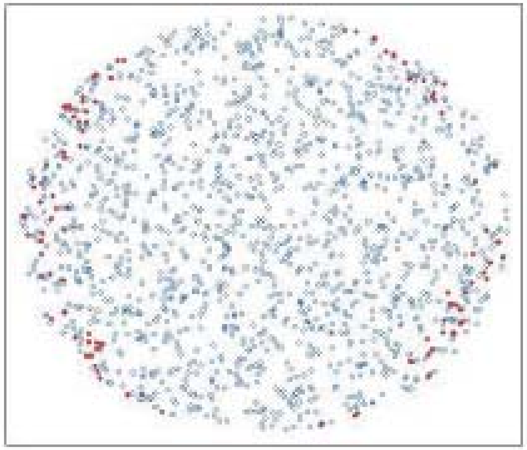


Figure 2: *Sky map simulation in hypertorus flat space (left). The fund polyhedron is a cube with length = 60 % the horizon size and contains 10 inal" sources (dark dots). One observes 1939 topological images (light d*

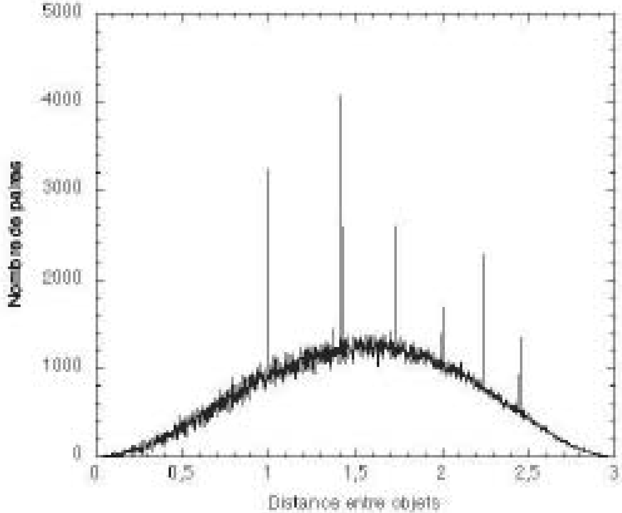


Figure 3: *The Pair Separation Histogram corresponding to Figure 2 exhibits spikes which stand out at values and with amplitudes depending on the topological properties of space.*

Test Cosmic Topology with CMB data : Two methods

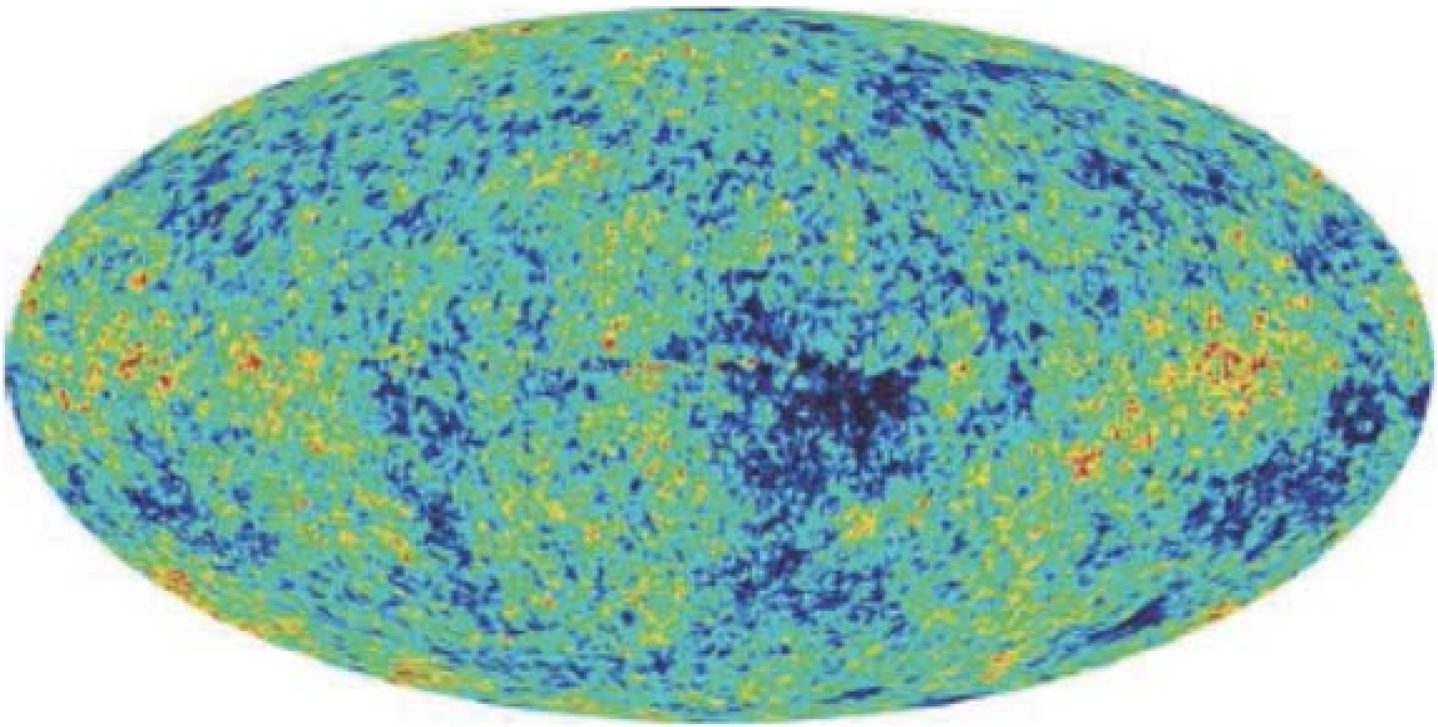
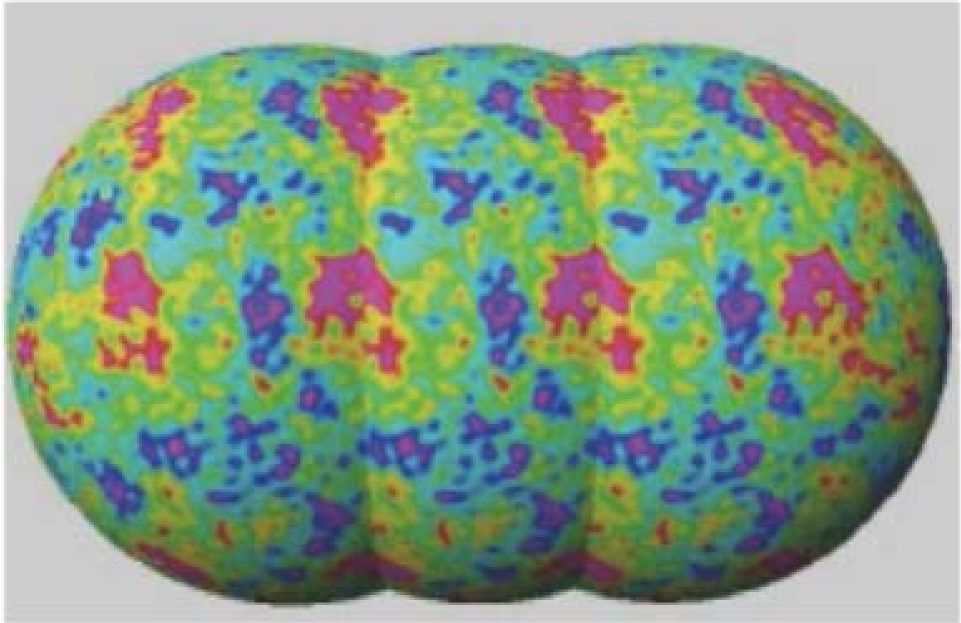


Figure 5: *Map of temperature anisotropies of CMB as observed by WMAP telescope. WMAP Homepage : http://map.gsfc.nasa.gov*

First method : Circles in the sky



Hypertorus -->
WMAP

Circles in the sky = Geometrical correlations on the LSS

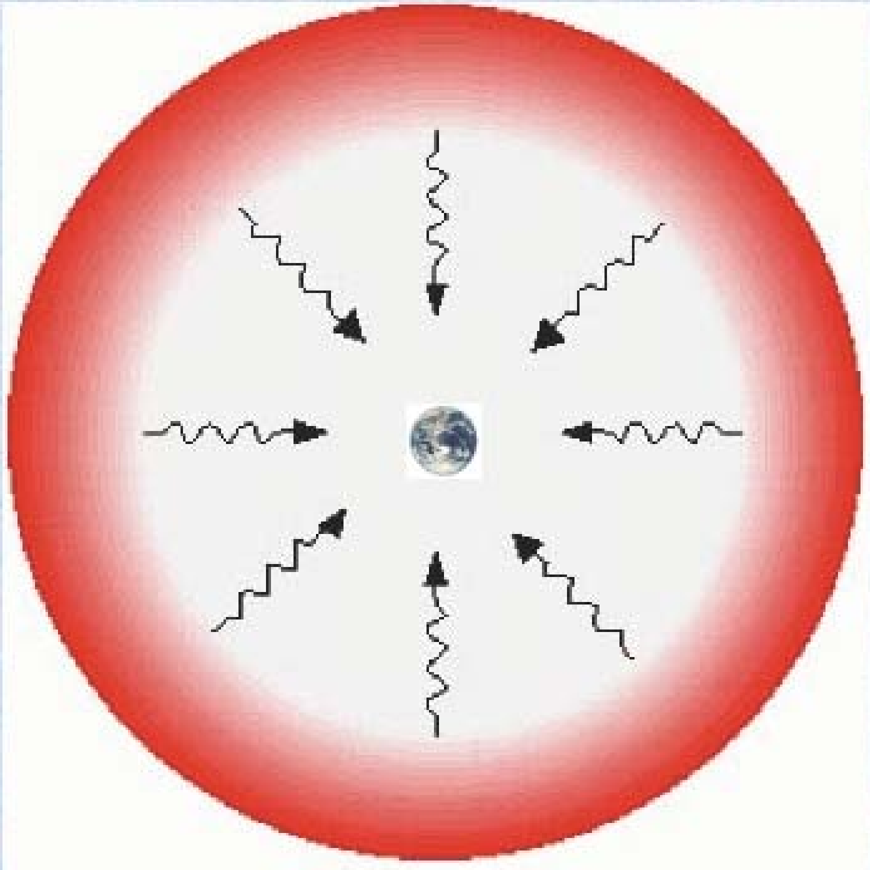
Difficult to exclude models, (some controversies)

Washed out by Doppler effect

→ ***not conclusive today***

Second methhod : Angular power spectrum

δT = function on a 2-sphere (=LSS)

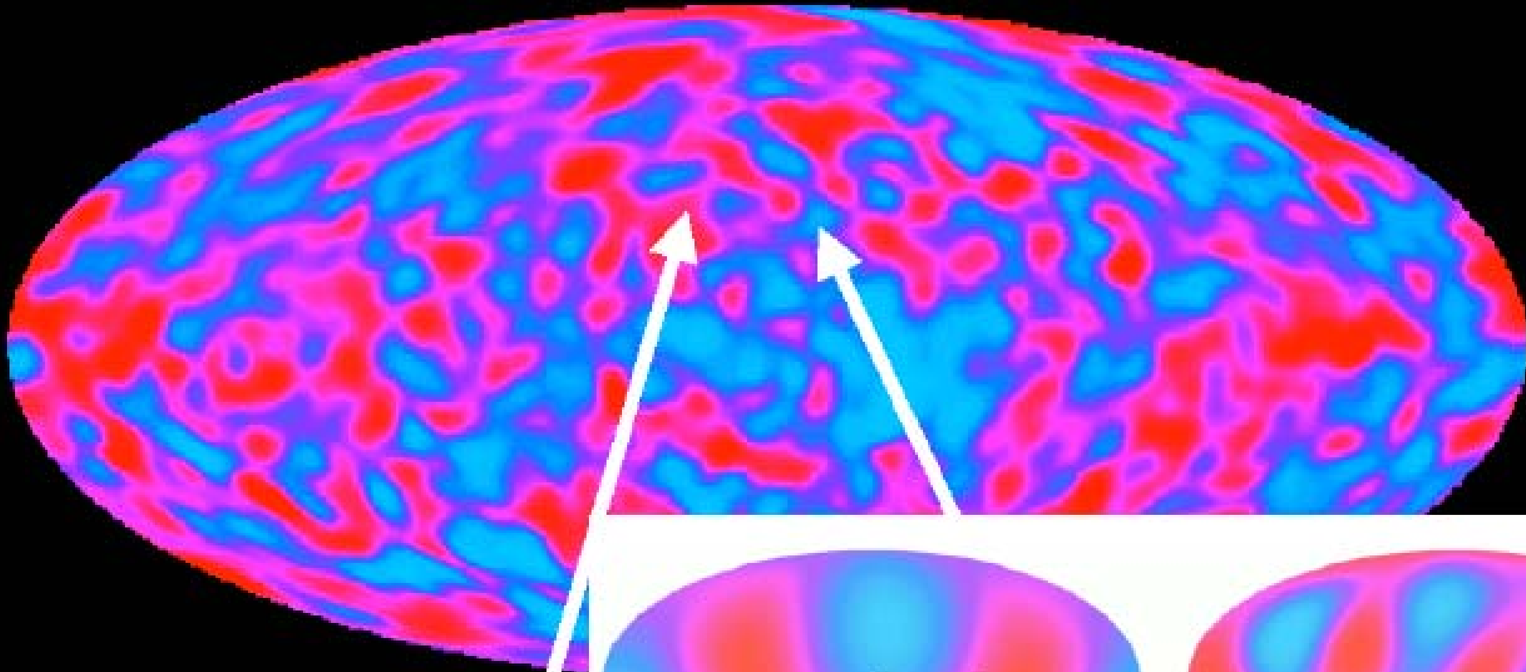


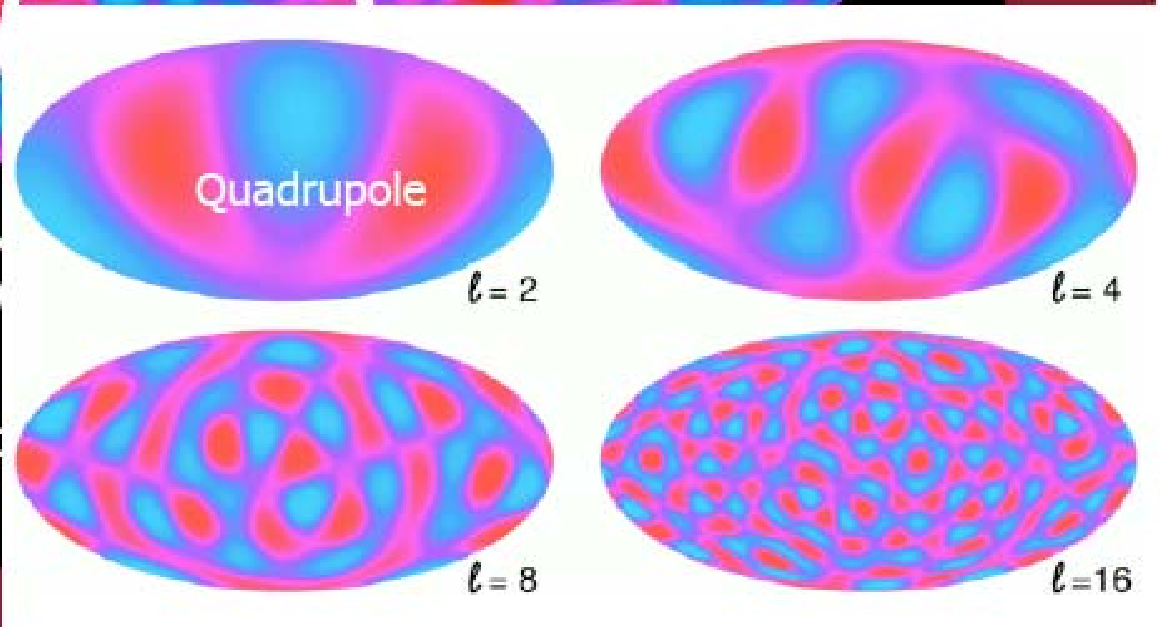
$$\delta T = \sum_{l} \sum_{m} a_{lm} Y_{lm}$$

$$C_l = \frac{1}{2l+1} \left\langle \sum_{-l}^{l} a_{lm}{}^2 \right\rangle$$

Multipole moments

Angular power spectrum, C_l





$T_1(\theta_1,$

$\langle T_1 T_2 \rangle =$

Motivations from WMAP power spectrum

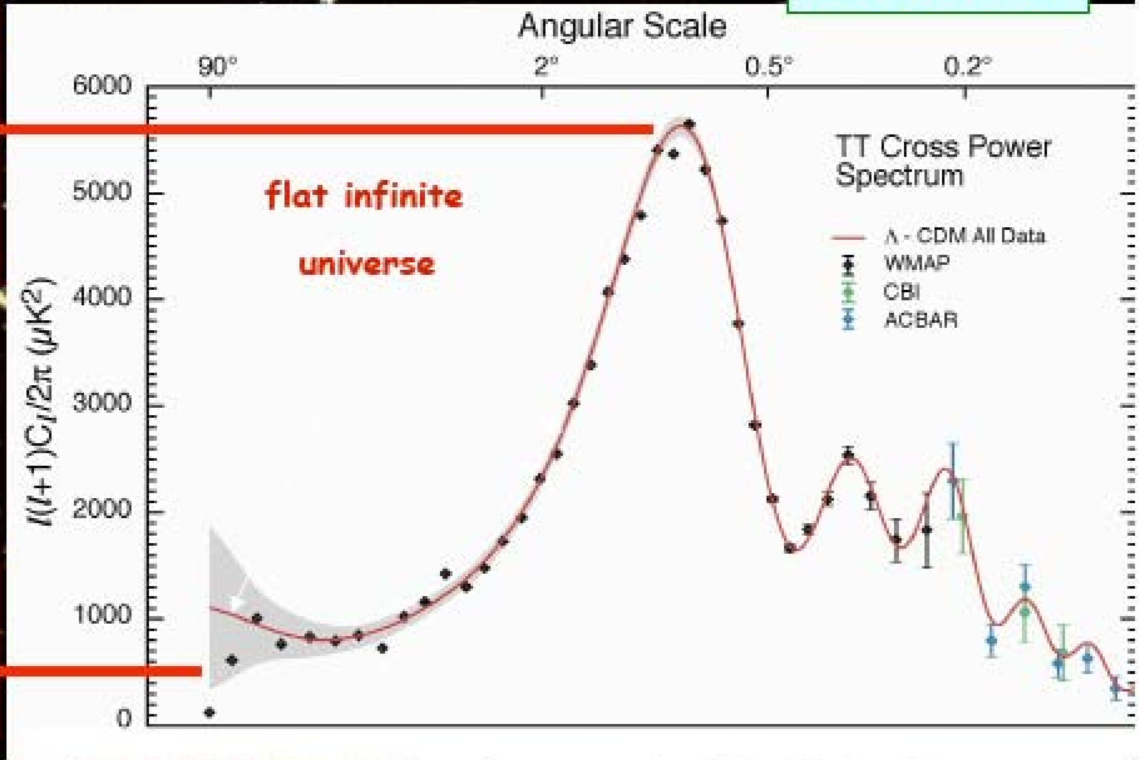
(Bennett et al., Spergel et al. 2003)

•Universe seems to be positively curved

$\Omega_{tot} = 1.02 \pm 0.02$ at $1\ \sigma$

• Lack of power at large scales (> 60°)

l=180°/θ

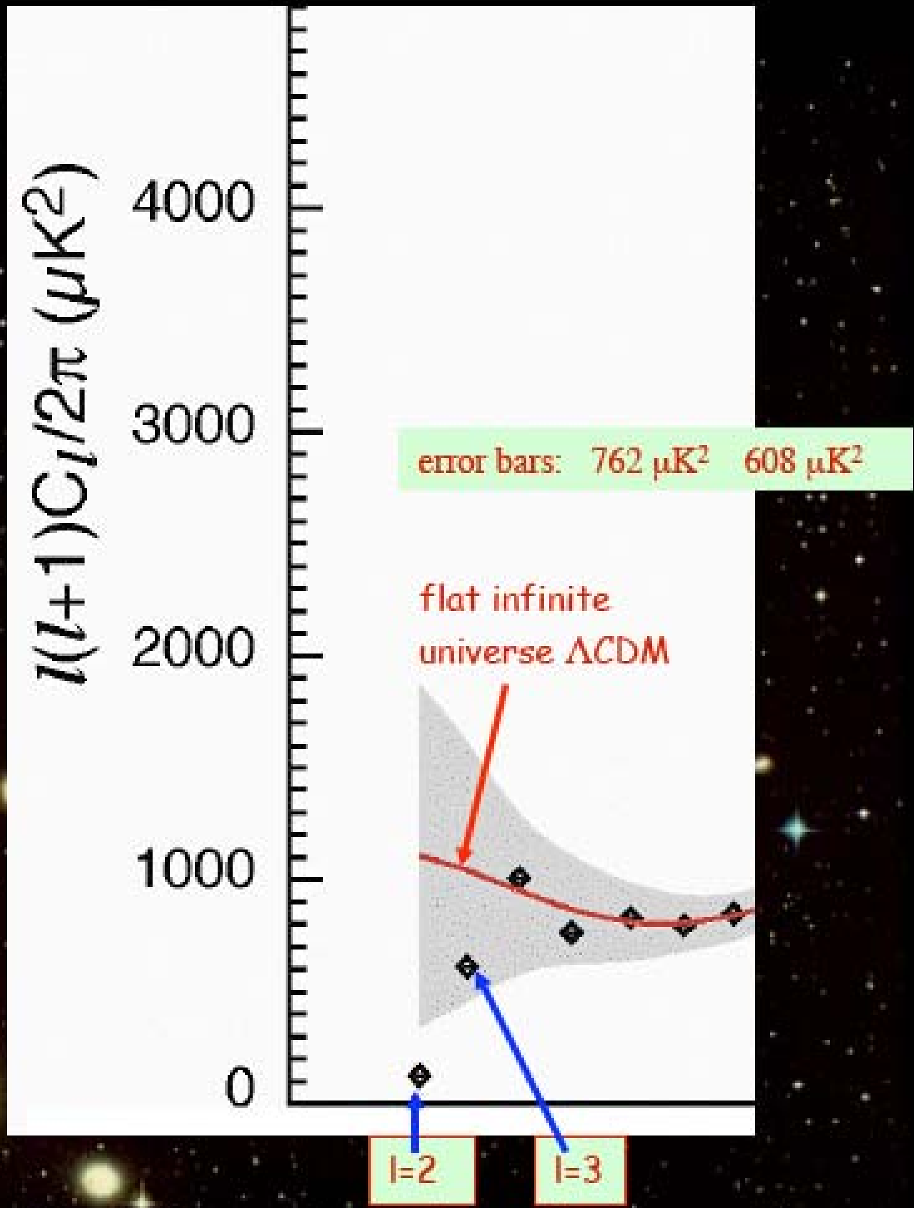


« Low l anomalies »

l = 2 quadrupole : 14%
l = 3 octopole : 72%

quadrupole plane and octopole planes aligned with local planes

→ Concordance model excluded at 99.95%!



Possible explanations

- Cosmic variance, bad data analysis ?
- Solar system effect ?(Schwarz et al. 2004; Hansen et al. 2004)

Primordial physics ?

- non trivial topology (MCM) ? :

• finite space cannot vibrate at scales larger than its size

→ reduction of power at low multipoles

• No isotropy → correlations in low multipoles

Origin of temperaturefluctuations

The δT fluctuations are imprinted on the LSS by the δρ fluctuations of matter (gravitational redshift, Doppler effect

$$\frac{\delta T}{T} = \frac{1}{4}\frac{\delta\rho}{\rho} + \Phi + v_b + \int (\dot{\Phi} - \dot{\Psi})\, dS$$

Density fluctuation +
Gravitational potential =
Sachs-Wolfe term

Line of-sight:
Integrated Sachs-Wolfe effect

Motion of plasma:
Doppler effect

Temperature fluctuations : Standard treatment

- Model the matter fluctuations $\delta\rho$

  (Gaussian statistics, power spectrum, cosmology)
- Calculate δT from $\delta\rho$

  (requires a code using Fourier modes of $\delta\rho$

  (or hyper spherical harmonics if curvature is positive)
- statistics of $\delta\rho \rightarrow$ stat of δT

(naturally expressed through spherical harmonics coefficients)

(= what is extracted from the measurements)

Note : if calculations provide δT under a different form, a conversion is required to compare with observations.

What changes in MCM ?

Same process but distribution of $\delta\rho$ with different properties

- No power on large spatial scales (do not exist in MCM !)
- Anisotropy in the distribution

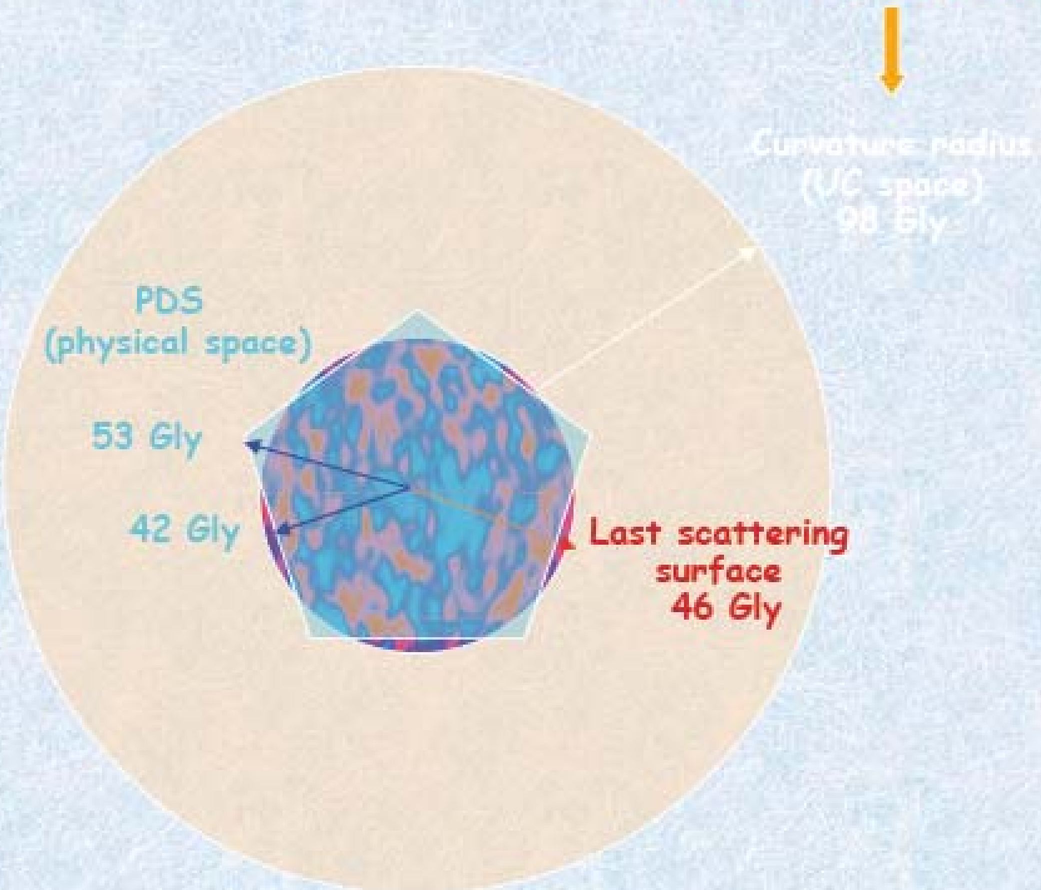
Otherwise, apply the same treatment.

Calculations require to know

the eigen-modes (of the Laplacian) of the space

Best fit PDS in « 2D »

(Ω_{tot} = 1.016, Ω_m =0.28)



Volume(Space) = 80 % Vol(U_{obs}) !

Cosmic lens !

Modes in multi-connected spaces

Modes of M = (uc) / G = Modes of (uc) which are G-invariant

(Ex. for the torus: Fourier mode --> periodic Fourier modes)
continuous distribution → discontinuous distribution

In most cases, harmonic modes are unknown.

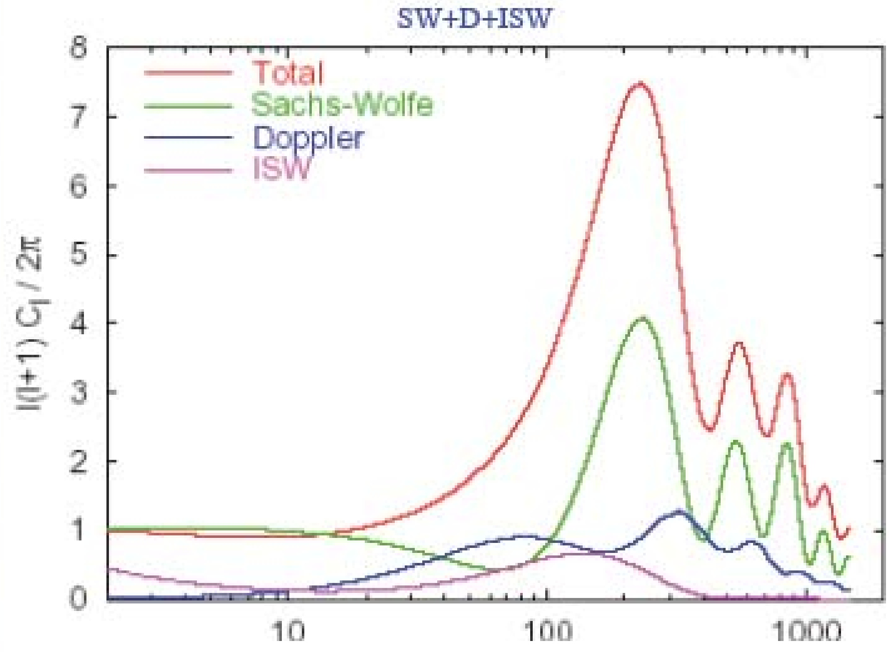
- first results : numerical calculations of a small number of eigenmodes.
- MLR et al : calculate the modes analytically

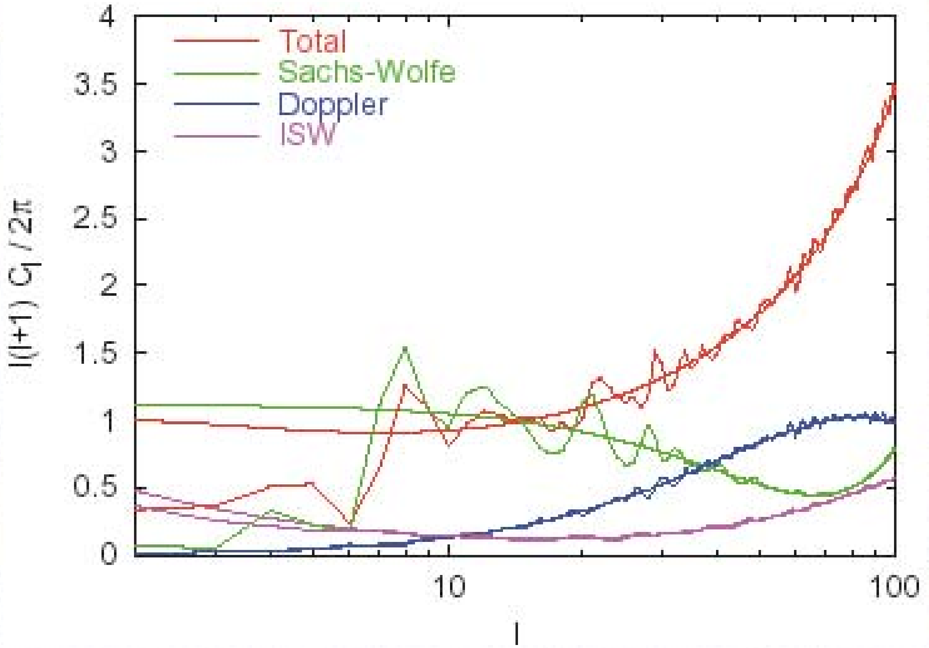
Modes in Poincaré Dodecahedral Spherical space (PDS)

Based on numerical calculations of a small number of eigenmodes.

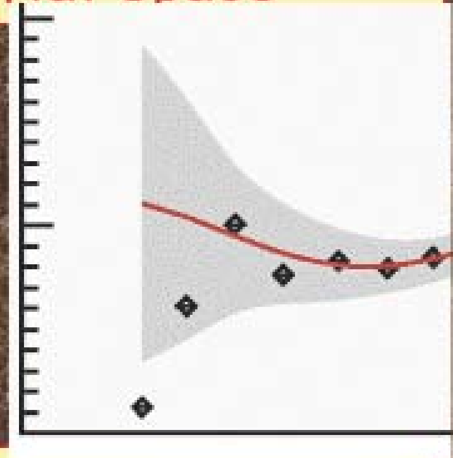
- fits low quadrupole
- fits low octopole
- $1.009 < \Omega_{tot} < 1.02$

*Luminet et al., Nature **425**, 593 (2003)*





Simply-connected flat space



Cut-off at large scale in cubic torus

RWULL: Phys.Rev.D69 (2004),

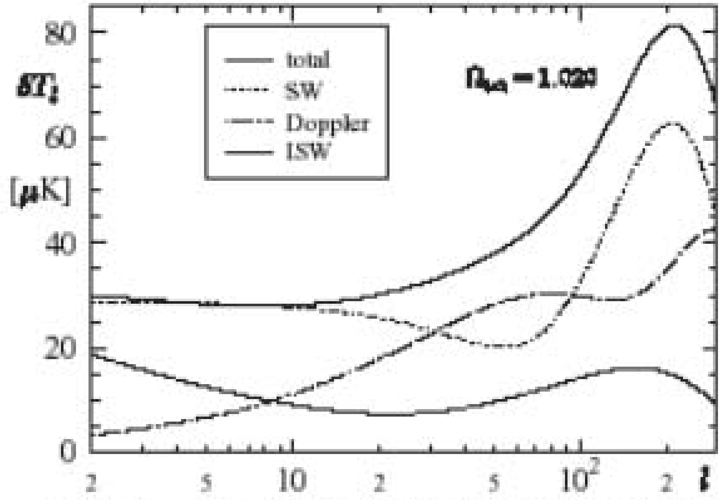


Figure 3. The angular power spectrum δT_l is shown for an S^3 model with $\Omega_{tot} = 1.020$ together with the ordinary Sachs-Wolfe (SW), the Doppler, and the integrated Sachs-Wolfe (ISW) contribution.

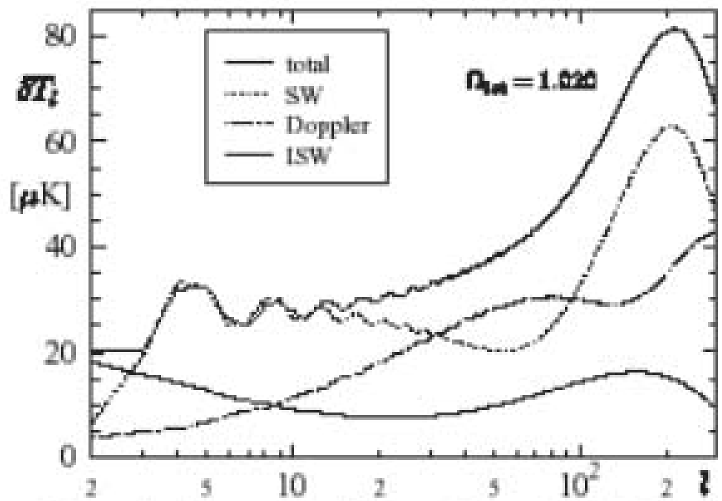
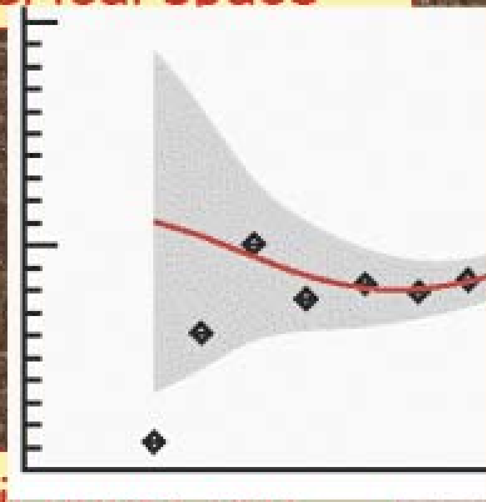


Figure 4. The same quantities are shown as in Fig. 3, but for the Poincaré dodecahedral space $\mathcal{D}$.

Simply-connected spherical space



Multiconnected spherical space (PDS)

Aurich et al. MNRAS (2005)

Calculations of eigen-modes of spherical space

Motivations : To express [the statistics of] the density fluctuations $\delta\rho$ at the epoch of recombination

→ temperature fluctuations $\delta T(\theta,\phi)$ in the CMB.

- $T(\theta,\phi)$, as a function on the sphere,
  naturally and conveniently expands in spherical harmonics Y_{lm}.

Calculations natural and easy (!) from the basis Y_{lm}

(numerical codes implemented with that basis)

Modes of [the Laplacian of] S^3

An eigenmode of space S^3 is a function solving the Helmoltz equation

$$\Delta F = \kappa F$$

- Ex. : hyperspherical harmonics Y_{klm}

for each k, vector space of dimension $(\kappa+1)^2$

Link easy with Y_{lm}

- Other example : - Wigner functions Z_{km1m2}

- intertwining : Clebsh-Gordan coefficients

Eigenmodes in spherical spaces $M = S^3/G$

A mode of S^3 / G is a mode of S^3, which is G-invariant.
→What are the G-invariant modes of S^3 (=uc) ?
ex. PDS : G = icosahedral group

Modes not kown in S^3 / G
First calculations only numeric, very limited !

→ Calculation of modes of S^3 / G (M.L.R.)

Case of PDS :
G = I = icosahedral group (60 symmetries of the icosahedron)
I = same group, reflexions included (120 elements)*

Spherical modes of PDS

- Two results allow analytical calculations
  (do not apply in the negative curvature case !)

  - SU(2) = S^3 as manifolds

  - SO(4) == SO(3) x SO(3) as groups;
    "single action transformations " involve only one of the two SO(3) factors

Modes of PDS

Three methods

1) use the quaternionic representation of S3 and the SO(4) action

2) apply group theory results to the Wigner functions

3) use Hopf fibration and Maxwell multipole vectors

Orbifold construction of the modes of the Poincaré dodecahedral space,

Lachièze-Rey M and Weeks J, 2008, Journal of Physics A: Math. Theor. 41 (2008) 295209 (http://hal.archives-ouvertes.fr/hal-00218221/fr/) (http://fr.arxiv.org/abs/0801.4232)

1 - Spherical modes with quaternions

Quaternions represent

- points of S^3,
- and also SO(4) actions

- *A new basis for eigenmodes on the Sphere*

  Lachieze-Rey M., Journal of Physics A, 37, issue 1, p205-210.
  http://fr.arxiv.org/abs/math.SP/0304409,
  http://www.iop.org/EJ/abstract/0305-4470/37/1/014/
- *Laplacian eigenmodes for the three-Sphere*,

Lachieze-Rey M. 2004, Journal of Physics A, 37, 21, p 5625-5634
(math.SP/0401153) (stacks.iop.org/0305-4470/37/5625)

2- Modes with Wigner functions

- Rather than hyperspherical harmonics Y_{klm},
  use the Wigner functions Z_{km1m2} as another basis

Write their transformations under SO(4)
and icosahedral group $I \subset SO(3) \subset SO(4)$.

- Select the invariant modes
- Convert to usual basis (intertwining involves Clebsch-Gordan coeff.)

- *Eigenmodes of Dodecahedral space*, Lachieze-Rey M. /\2004, CQG, vol.21, 9, p 2455-2464(gr-qc/0402035)
- *Laplacan eigenmodes for spherical spaces*, M. Lachieze-Rey and S. Caillerie 2005, Class. Quantum Grav. 22 (2005) 695-708.(astro-ph/0501419)

3- Spherical modes from Hopf fibration : (with G Weeks)

•1) Using Hopf fibration S^3 --> S^2 :

• An S^2 L-mode Y_L generates a vertical S^3 K-modes $\mathcal{Y}_K$ (K=2 L)

(* vertical := ct on the fibers)

• 2) G*-invariance of $\mathcal{Y}_K$ ↔ G-invariance of Y_L

→ each G-invariant Y_L generates a vertical spherical mode $\mathcal{Y}_K$

→One solution !

• 3) Search G-invariant Y_L using polar decomposition -->

a spherical L-harmonics <--> (modulo a constant),

L directions in R^3 = L points in projective space PR^3.

a G-invariant spherical L-harmonics <--> (modulo a constant),

a G-invariant set of L points in PR^3.

Hopf fibration

modes S^3 from modes S^2

Using Hopf fibration S^3 --> S^2 :

- 1) Any S^2 ℓ-mode Y_ℓ generates a vertical S^3 k-modes $\mathcal{Y}_K$ ($k=2\ell$)

  (* vertical := ct on the fibers)

- 2) on S3, any vertical k-modes $\mathcal{Y}_K$ generates (k+1) k-modes $\mathcal{Y}_K$

- 3) The totality of the $(k+1)^2$ k-modes is generated from the ($2\ell+1 = k+1$) $Y_{\ell m}$

G*-invariance of $\mathcal{Y}_K$ ←→ G-invariance of Y_L

→ each G-invariant Y_L generates a vertical spherical mode $\mathcal{Y}_K$

→One solution !

- 3) Search G-invariant Y_L using polar decomposition -->

a spherical L-harmonics <--> (modulo a constant),

L directions in R^3 = L points in projective space PR^3.

a G-invariant spherical L-harmonics <--> (modulo a constant),

a G-invariant set of L points in PR^3.

$$
\begin{array}{ccccccccccc}
\mathcal{Y}_{k,+\ell,-k/2} & \leftarrow & \cdots & \leftarrow & \mathcal{Y}_{k,+\ell,-1} & \leftarrow & \mathcal{Y}_{k,+\ell,0} & \rightarrow & \mathcal{Y}_{k,+\ell,+1} & \rightarrow & \cdots & \rightarrow & \mathcal{Y}_{k,+\ell,+k/2} \\
 & & & & & & \Uparrow & & & & & & \\
 & & & & & & Y_{\ell,+\ell} & & & & & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\mathcal{Y}_{k,+1,-k/2} & \leftarrow & \cdots & \leftarrow & \mathcal{Y}_{k,+1,-1} & \leftarrow & \mathcal{Y}_{k,+1,0} & \rightarrow & \mathcal{Y}_{k,+1,+1} & \rightarrow & \cdots & \rightarrow & \mathcal{Y}_{k,+1,+k/2} \\
 & & & & & & \Uparrow & & & & & & \\
 & & & & & & Y_{\ell,+1} & & & & & & \\
\mathcal{Y}_{k,0,-k/2} & \leftarrow & \cdots & \leftarrow & \mathcal{Y}_{k,0,-1} & \leftarrow & \mathcal{Y}_{k,0,0} & \rightarrow & \mathcal{Y}_{k,0,+1} & \rightarrow & \cdots & \rightarrow & \mathcal{Y}_{k,0,+k/2} \\
 & & & & & & \Uparrow & & & & & & \\
 & & & & & & Y_{\ell,0} & & & & & & \\
\mathcal{Y}_{k,-1,-k/2} & \leftarrow & \cdots & \leftarrow & \mathcal{Y}_{k,-1,-1} & \leftarrow & \mathcal{Y}_{k,-1,0} & \rightarrow & \mathcal{Y}_{k,-1,+1} & \rightarrow & \cdots & \rightarrow & \mathcal{Y}_{k,-1,+k/2} \\
 & & & & & & \Uparrow & & & & & & \\
 & & & & & & Y_{\ell,-1} & & & & & & \\
\vdots & & & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\mathcal{Y}_{k,-\ell,-k/2} & \leftarrow & \cdots & \leftarrow & \mathcal{Y}_{k,-\ell,-1} & \leftarrow & \mathcal{Y}_{k,-\ell,0} & \rightarrow & \mathcal{Y}_{k,-\ell,+1} & \rightarrow & \cdots & \rightarrow & \mathcal{Y}_{k,-\ell,+k/2} \\
 & & & & & & \Uparrow & & & & & & \\
 & & & & & & Y_{\ell,-\ell} & & & & & &
\end{array}
$$

modes S^3/I^* from modes S^2/I

G*-invariance of $\mathcal{Y}_K$ ←→ G-invariance of Y_L

→ each G-invariant Y_ℓ generates (k+1) spherical modes $\mathcal{Y}_K$

→ each S^2/I mode generates (k+1) S3/I* modes

→Find the PDS lmodes = find the S^2/I modes

Search the the S^2/I modes (=I-invariant S2 modes) using polar decomposition

Maxwell polar decomposition of S^2 modes

Any function on S^2 (eg , T_{CMB}): $F = \Sigma_\ell F_\ell$

- spherical harmonics : $F_\ell = \Sigma_{\ell m} F_{\ell m} Y_{\ell m}$: $F_\ell \Leftrightarrow (F_{\ell 1}, \ldots, F_{\ell\ell})$

- Maxwell's multipole vector decomposition ($r = (x^2 + y^2 + z^2)^{1/2}$)

$$F_\ell = C^{te} \; r^{2\ell+1} \; \nabla_{v\ell} \cdots \nabla_{v2} \nabla_{v1} (1/r)$$

$$F_\ell \Leftrightarrow \{d_1, \ldots, d_\ell\}$$

(up to Ct)
(up to flipping the signs) (the ordering of the directions is irrelevant)

I-invariant S^2 modes

f is uniquely represented by $\{d_1, \ldots, d_\ell\}$

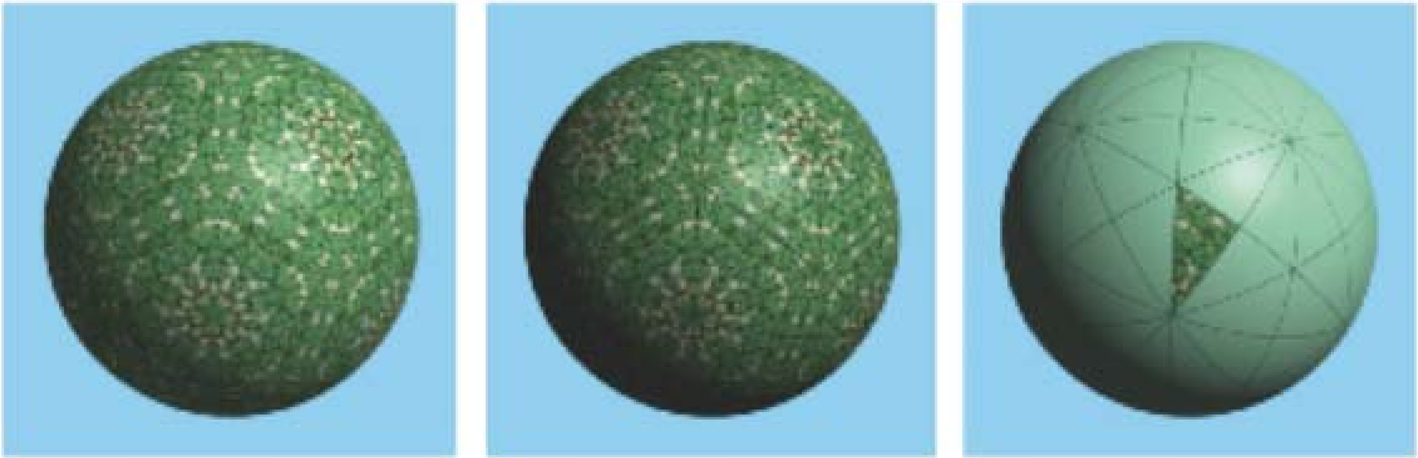
f I-invariant <==> the set $\{d_1, \ldots, d_\ell\}$ is I-invariant

the d_1 live in the orbifold S_2/G

check = one finds the good multiplicity of modes (Ikeda)
--> one finds the modes in terms of their Maxwell decomposition !

Orbifold

Orbifold S^2/I



(*a*) (*b*) (*c*)

Figure 2. How to construct the *235 orbifold. (*a*) Begin with an icosahedrally symmetric pattern on the 2-sphere. (*b*) Locate all lines of mirror symmetry. Each is a great circle, and together they divide the sphere into 120 congruent triangles. (*c*) Fold the sphere along all mirror lines simultaneously, so that the whole sphere maps 120-to-1 onto a single triangle. The resulting quotient is the *235 orbifold. The Conway notation *235 may be understood as follows: the '*' denotes the mirror-symmetric origin of the triangle's sides, while the 2, the 3 and the 5 denote the fact that 2, 3 and 5 mirror lines met at each corner, respectively.

Lachièze-Rey, 2008

• Each point in the *interior of the triangle* : *whole point*
← → *invariant set of 120 points on* S^2
← → *invariant set of 60 directions*.

• *each point on a mirror boundary*
← → *30 directions : 'half point'*.

• *A point at the corner reflector of angle* $\pi/2$
← → *15 directions, : 'quarter point'*.
• *points at the corner reflectors of angle* $\pi/3$ *and* $\pi/5$ *: 1/6 point and a 1/10 point*.

$C_{1/10}$ *denote the number of 1/10 points*
$C_{1/6}$ *denote the number of 1/6 points*
$C_{1/4}$ *denote the number of 1/4 points*
$C_{1/2}$ *denote the number of half points*
C_1 *denote the number of whole points*

space V_ℓ *of I-invariant* ℓ*-eigenmodes of* S^2
= *space* V_ℓ *of I* ℓ*-eigenmodes of* S^2/I

dimension $= 1 + C_{1/2} + 2\,C_1 = 1 + [\ell/2] + [\ell/3] + [\ell/5] - \ell.$

➔ *space of k-modes of the PDS* $S^3/I*$
Dimension $= (k+1)\,([1 + [k/4] + [k/6] + [k/10] - k/2)$.

This gives Ikeda's formula !

Ikeda

THEOREM 4.6. (1) *The spectrum of Laplacian for the homogeneous spherical space form* S^3/I^* *is*

eigenvalue	*multiplicity*
$4k(k+1)$	$(2k+1)(1+[k/5]+[k/3]+[k/2]-k)$

where k *runs through all nonnegative integers except* 1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23, 29. (2) *Eigenvalues of Laplacian for the homogeneous spherical*

We know the mode in the form of multipole vectors.
(Maxwell decomposition)

For practical calculations, one has to convert
$(v1 \ldots v_l) \rightarrow Y_{lm} \rightarrow Y_{klm}$