Strong Binary Pulsar Constraints on Lorentz Violation in Gravity

Enrico Barausse

(Institut d'Astrophysique de Paris/CNRS)

based on

Kent Yagi, Diego Blas, Nicolas Yunes, EB, arXiv:1307.6219

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Outline

- Lorentz violation in gravity: why and how

  Einstein aether theory and khronometric theory (= low-energy Horava gravity)

- Lorentz violation implies violations of the strong equivalence principle:

  The motion of neutron stars (the “sensitivities” and dipolar gravitational-wave emission)

- Constraints from binaries containing pulsar

Lorentz violation in gravity: why?

- LV may give better UV behavior (Horava), QG completions generally lead to LV
- LV allows MOND-like (Bekenstein, Blanchet & Marsat) or DE-like phenomenology
- Strong constraints in matter sector, weaker ones in gravity sector (caveat: constraints expected to percolate from gravity to matter sector)
- Solar system/isolated & binary pulsar experiments historically used to constrains LV in weak field (1 PN) regimes ("preferred-frame parameters": Nordvedt, Kramer, Wex, Damour, Esposito Farese), but surprises may happen in stronger-field regimes

Einstein-aether theory

- We want to specify a (local) preferred time "direction"

  → timelike aether field U_μ with unit norm

- Most generic action (in 4D) that's quadratic in derivatives is given (up to total derivatives) by

$$S_{æ} = \frac{1}{16\pi G_{æ}} \int d^4x \sqrt{-g} \left(-R - M^{\alpha\beta}{}_{\mu\nu} \nabla_\alpha U^\mu \nabla_\beta U^\nu\right)$$

$$M^{\alpha\beta}{}_{\mu\nu} = c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta^\alpha_\mu \delta^\beta_\nu + c_3 \delta^\alpha_\nu \delta^\beta_\mu + c_4 U^\alpha U^\beta g_{\mu\nu}$$

- To satisfy weak equivalence principle, matter fields couple minimally to metric (and not directly to aether)

$$S = S_{æ} + S_{\text{matter}}(\psi, g_{\mu\nu})$$

Khronometric gravity

- To specify a global time, U must be hypersurface orthogonal ("khronometric" theory)

$$U_\mu = \frac{\partial_\mu T}{\sqrt{g^{\mu\nu}\partial_\mu T \partial_\nu T}} \qquad S_{æ} = \frac{1}{16\pi G_{æ}} \int d^4x \sqrt{-g}\,\left(-R - M^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha U^\mu \nabla_\beta U^\nu\right)$$

- Because *U* is timelike, *T* can be used to as time coordinate

$$U_\alpha = \delta^T_\alpha (g^{TT})^{-1/2} = N\delta^T_\alpha \qquad a_i = \partial_i \ln N$$

$$S_K = \frac{1}{16\pi G_K}\int dT d^3x\, N\sqrt{h}\,(K_{ij}K^{ij} - \mu K^2 \; + \xi\, {}^{(3)}R + \eta a_i a^i)$$

- 3 free parameters vs 4 of AE theory (because aether is hypersurface orthogonal)

Khronometric vs Horava gravity

$$S_H = \frac{1}{16\pi G_K} \int dT d^3x \, N\sqrt{h} \left(L_2 + \frac{1}{M_\star^2} L_4 + \frac{1}{M_\star^4} L_6 \right)$$

$$L_2 = K_{ij}K^{ij} - \mu K^2 + \xi^{(3)}R + \eta a_i a^i,$$

- L_4 and L_6 contain 4th- and 6th-order terms in the spatial derivatives
- Lower bound on M* depends on details of percolation of Lorentz violations from gravity to matter: from Lorentz violations in gravity alone, $M_\star \gtrsim 10^{-3}\ \mathrm{eV}$, but precise bounds depend on percolation
- Theory remains perturbative at all scales if $M_\star \lesssim 10^{16}\ \mathrm{GeV}$
- Terms crucial in the UV, but unimportant astrophysically, ie error scales as $\sim M_{\rm Planck}^4/(M M_\star)^2 \sim 10^{-14}(M_\odot/M)^2$

Constraints on the coupling constants: the Parametrized Post-Newtonian expansion

- At 1PN, theories = to GR except for preferred-frame parameters α_1 and α_2 which are zero in GR but not in LV gravity
- Solar system & pulsar experiments require $|\alpha_1| \lesssim 10^{-5}$ $|\alpha_2| \lesssim 10^{-9}$
- Imposing $\alpha_1 = \alpha_2 = 0$ reduces couplings from 4 to 2 (AE theory) ….

$$c_\pm = c_1 \pm c_3$$

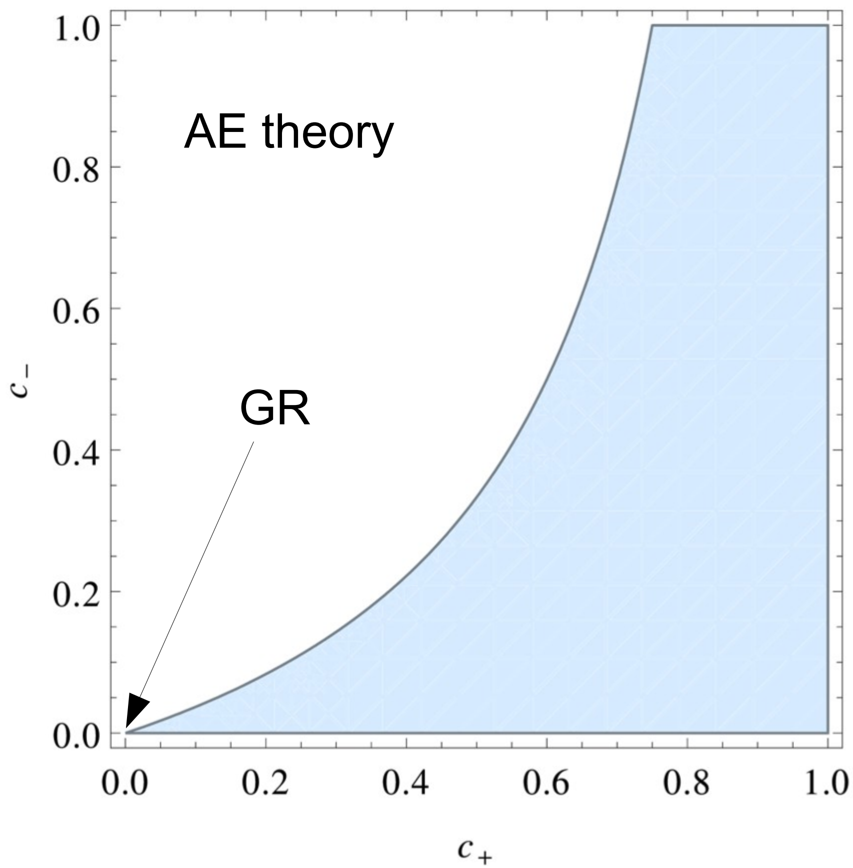
… and from 3 to 2 (khronometric theory) $\lambda = \mu/\xi - 1\,, \quad \beta = (\xi - 1)/\xi$

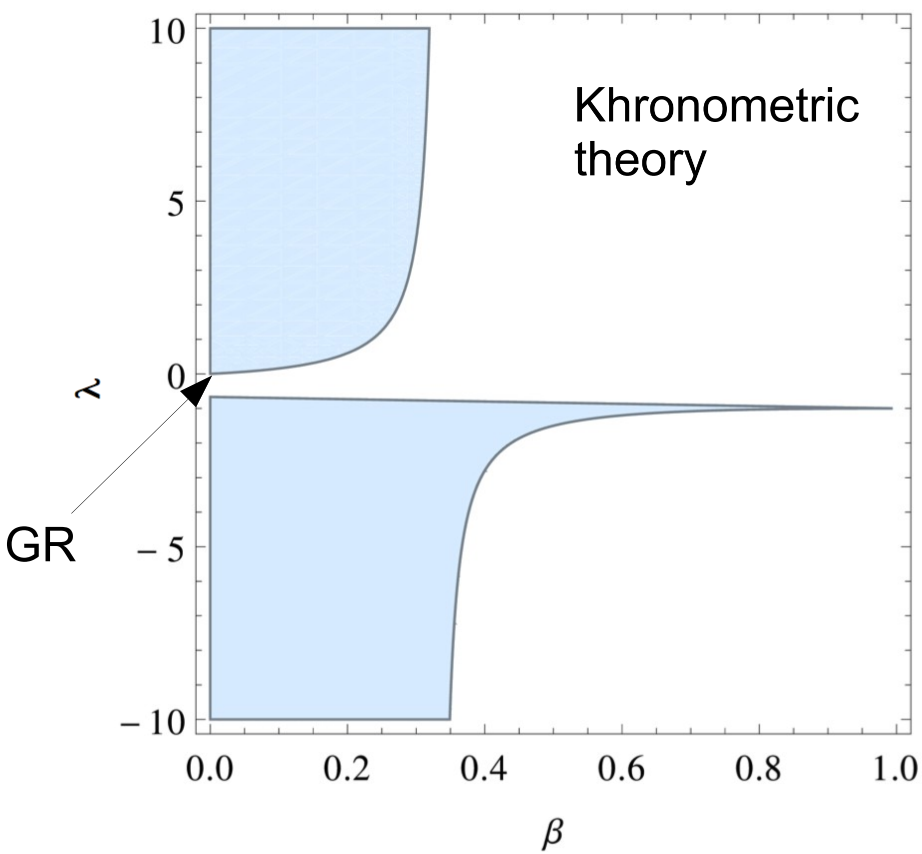
- c_+, c_- and λ, β enter at PN order > 1 (they are "stronger-field")

Constraints on the coupling constants: stability

- AE theory has propagating spin-0, spin-1 and spin-2 gravitational modes
- Khronometric theory has spin-0, spin-2 modes
- For stability, propagation speeds need to be real (no tachyonic instability)
- Propagation speed must be larger than speed of light to avoid gravitational Cherenkov radiation

Stability+Cherenkov constraints

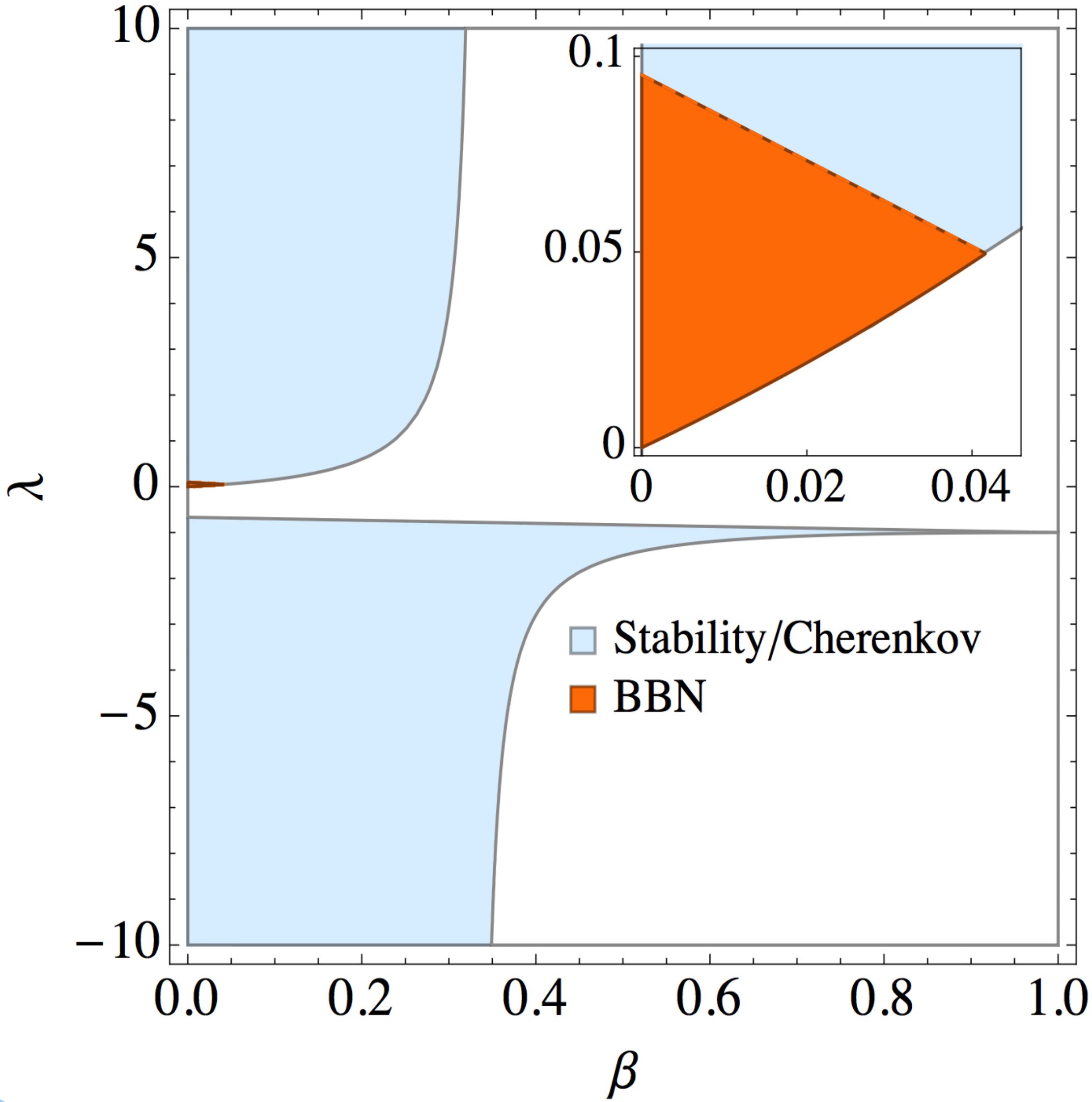




$$c_\pm = c_1 \pm c_3$$

$$\lambda = \mu/\xi - 1\,, \quad \beta = (\xi - 1)/\xi$$

How about cosmological constraints?



- Weak for AE theory
- For khronometric theory,

$$\frac{G_N}{G_C} = \frac{2+\beta+3\lambda}{2(1-\beta)}$$

and BBN requires

$$|G_N/G_C - 1| < 1/8$$

- No constraints from CMB in khronometric theory yet

Why are astrophysical effects expected?

- Matter couples minimally to metric, but metric couples non-minimally to aether $\Rightarrow$ effective matter-aether coupling in strong-field regimes
- For strongly gravitating body (e.g. neutron star), binding energy depends on velocity relative to the aether $\gamma = U_\mu u^\mu$ (i.e. structure depends on motion relative to preferred frame, as expected from Lorentz violation!)
- Gravitational mass depends on velocity relative to the aether

$$\Rightarrow \quad S_{matter} = \Sigma_i \int m_i(\gamma)\, d\tau_i \quad \Rightarrow \quad u_a^\mu \nabla_\mu (m_a u^\nu) = -\frac{d m_a}{d\gamma} u^\mu \nabla^\nu U_\mu$$

Violations of strong equivalence principle (aka Nordtvedt effect in Brans Dicke theory, scalar tensor theories, etc)

Why are astrophysical effects expected?

Whenever strong equivalence principle (SEP) is violated, dipolar gravitational-wave emission may be produced

- In GR, dipolar emission not present because of SEP + conservation of linear momentum

$$M_1 \equiv \int \rho\, x_i\, d^3x \qquad h \sim \frac{G}{c^3}\frac{d}{dt}\frac{M_1}{r} \sim \frac{G}{c^3}\frac{P}{r}$$

not a wave!

- If SEP is violated,

$$h \sim \frac{1}{R}\frac{d}{dt}\left[m_1(\gamma)x_1 + m_2(\gamma)x_2\right] \sim \frac{\mu}{R} v \left(\frac{d\log m_1}{d\log\gamma} - \frac{d\log m_2}{d\log\gamma}\right)$$

- Dipolar mode might be observable directly by interferometers, or indirectly via its backreaction on a binary's evolution

A PN analysis: the violation of the SEP

$$S_A = -\int d\tau\, \tilde{m}_A[\gamma] = -\tilde{m}_A \int d\tau\, \left\{1 + \sigma_A (1-\gamma_A) + \frac{1}{2}\sigma'_A (1-\gamma_A)^2 + \mathcal{O}\left[(1-\gamma_A)^3\right]\right\}$$

$$\gamma = U^\mu u_\mu \qquad \sigma_A \equiv -\left.\frac{d\ln \tilde{m}_A[\gamma_A]}{d\ln \gamma_A}\right|_{\gamma_A=1} \qquad \sigma'_A \equiv \sigma_A + \sigma_A^2 + \left.\frac{d^2 \ln \tilde{m}_A[\gamma_A]}{d\ln \gamma_A^2}\right|_{\gamma_A=1}$$

body's "sensitivities"

Define "active" gravitational mass $m_A = (1+\sigma_A)\tilde{m}_A$

and "strong-field" gravitational constant $\mathcal{G}_{AB} = \dfrac{G_N}{(1+\sigma_A)(1+\sigma_B)}$

Modified Newton's law:

$$\dot{v}^i_A = \sum_{B\neq A} \frac{-G_N \tilde{m}_B}{(1+\sigma_A) r^3_{AB}} r^i_{AB} \equiv \sum_{B\neq A} \frac{-\mathcal{G}_{AB} m_B}{r^3_{AB}} r^i_{AB}$$

Foster 2007

A PN analysis: the dissipative dynamics

- GWs carry energy away from binaries

$$S=(s_1 m_2+s_2 m_1)/M$$
$$M=m_1+m_2$$
$$\mu=\frac{m_1 m_2}{M}, \quad s_A=\sigma_A/(1+\sigma_A)$$

$$\dot{\mathcal{E}} = -\frac{32}{5} G_N (G_N M)^{4/3} \mu^2 \left(\frac{P_b}{2\pi}\right)^{-10/3} \langle \mathcal{A} \rangle$$

$$\langle \mathcal{A} \rangle = \frac{1}{(1+\sigma_1)^{4/3}(1+\sigma_2)^{4/3}} \left[\mathcal{A}_1 + \mathcal{S}\mathcal{A}_2 + \mathcal{S}^2 \mathcal{A}_3 \right.$$

→ Quadrupole

$$\left. + \frac{5}{32}(s_1 - s_2)^2 \mathcal{C} (1+\sigma_1)^{2/3}(1+\sigma_2)^{2/3} \left(\frac{P_b}{2\pi G_N M}\right)^{2/3} \right]$$

→ Dipole

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are functions of the coupling constants (c_+, c_-) or (β, λ);

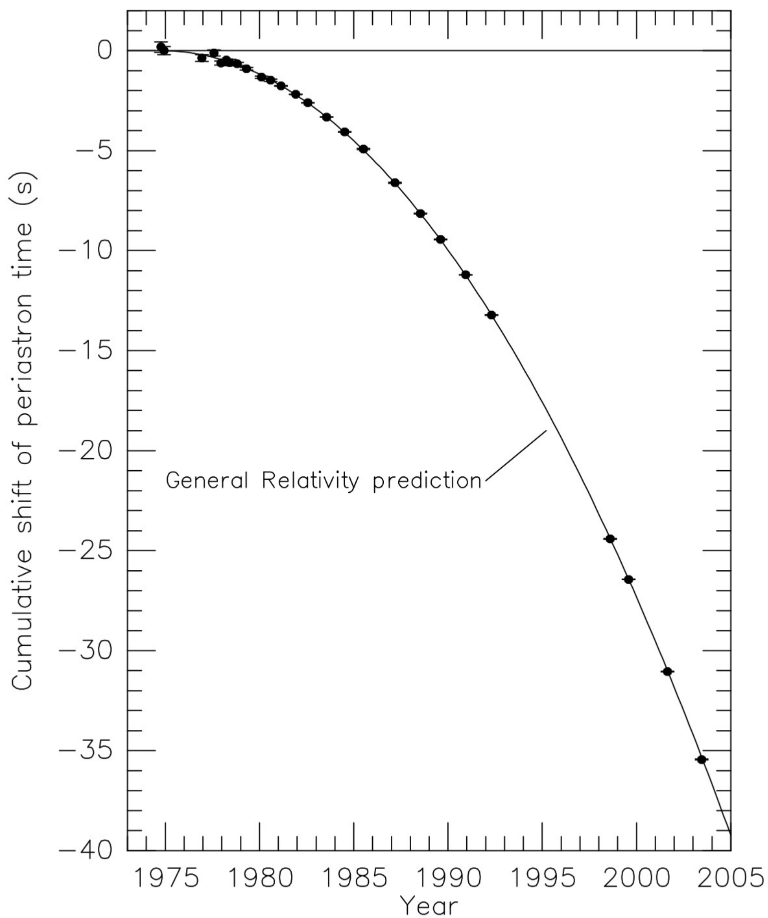
in GR $\mathcal{A} = 1$ (Foster 2007, Yagi, Blas, Yunes, EB 2013)

- As binary's binding energy decreases, period decreases

$$\frac{\dot{P}_b}{P_b} = -\frac{3}{2}\frac{\dot{E}}{E} = \frac{3}{2}\frac{\dot{\mathcal{E}}}{E}$$

Why is this interesting?

Binary pulsars are the strongest test of GR to date



To calculate rate of change of orbital period we need sensitivities

$$\sigma = -\left.\frac{\partial \log M}{\partial \log \gamma}\right|_{v=0} = -2\left.\frac{\partial \log M}{\partial (v^2)}\right|_{v=0}$$

PSR B1913+16
(Weisberg & Taylor 2004)

A strong-field derivation of the sensitivities

- Consider stationary configuration descriving NS and aether moving with small velocity *v* against it, i.e. asymptotically

$$
\begin{aligned}
ds^2 = \Bigg[& \left(1-\frac{2M_*}{r}\right)dt^2 - \left(1+\frac{2M_*}{r}\right)dr^2 \\
& -r^2 d\theta^2 - r^2\sin^2\theta d\phi^2 \\
& -2v(B^- + B^+ + 4)\frac{M_*}{r}\cos\theta dt dr \\
& +v(7+2B^-)M_*\sin\theta dt d\theta \Bigg] \times \left[1+\mathcal{O}\left(v,\frac{1}{r}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
U_\mu dx^\mu = \Bigg\{ & \left(1-\frac{M_*}{r}\right)dt \\
& + v\left[1-(1+B^- + B^+ + C^- + C^+)\frac{M_*}{r}\right]\cos\theta dr \\
& - vr\left[1-(3+2B^- + 2C^-)\frac{M_*}{2r}\right]\sin\theta d\theta \Bigg\} \\
& \times \left[1+\mathcal{O}\left(v,\frac{1}{r}\right)\right],
\end{aligned}
$$

- All fields are time-independent, so $M = -\int_\Sigma d^3x(\mathcal{L}_g + \mathcal{L}_U + \mathcal{L}_m)$

$$\mathcal{L}_g = \sqrt{-g}g^{\mu\nu}(\Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\lambda\alpha}) \qquad \mathcal{L}_U = -\frac{1}{16\pi G}\sqrt{-g}M^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha U^\mu \nabla_\beta U^\nu$$

A strong-field derivation of the sensitivities

- Taking difference between configurations with v and $v+\delta v$, bulk terms disappear (because we are on shell).
- Surface terms from metric vanish because of boundary conditions, and those from matter vanish because no matter at spatial infinity

$$\delta M = -\int_{\partial\Sigma} d^2 S_i \delta U^\mu \left(\frac{\partial \mathcal{L}_U}{\partial(\partial_i U^\mu)}\right)$$

- Surface integral can be evaluate asymptotically to get

$$\frac{\partial \log M}{\partial v}(v) = -\bar{\sigma}_{Æ}\, v \qquad \bar{\sigma}_{Æ} = -\frac{2c_1\left[2\left(B^+ + B^-\right) + 8 + \alpha_1\right]}{\left(c_1 - c_3\right)\left(8 + \alpha_1\right)}$$

$$\Rightarrow \quad \sigma = -\left.\frac{\partial^2 \log M}{\partial v^2}\right|_{v=0} = \bar{\sigma}_{Æ}$$

To get sensitivity, we only need slowly-moving NS solution!

The PN way to the sensitivities

Solving the field equations in the standard PPN gauge for a system of bodies with velocities v_A relative to the aether (which is asymptotically at rest):

$$g_{00} = -1 + 2\sum_A \frac{G_N \tilde{m}_A}{r_A} - 2\sum_{A,B} \frac{G_N^2 \tilde{m}_A \tilde{m}_B}{r_A r_B} - 2\sum_{A,B\neq A} \frac{G_N^2 \tilde{m}_A \tilde{m}_B}{r_A r_{AB}} + 3\sum_A \frac{G_N \tilde{m}_A}{r_A} v_A^2 (1+\sigma_A),$$

$$g_{ij} = \left(1 + 2\sum_A \frac{G_N \tilde{m}_A}{r_A}\right)\delta_{ij},$$

$$g_{0i} = \sum_A B_A^- \frac{G_N \tilde{m}_A}{r_A} v_A^i + \sum_A B_A^+ \frac{G_N \tilde{m}_A}{r_A^3} (v_A^j r_A^j) r_A^i,$$

$$u^0 = 1 + \sum_A \frac{G_N \tilde{m}_A}{r_A},$$

$$u^i = \sum_A C_A^- \frac{G_N \tilde{m}_A}{r_A} (v_A)^i + \sum_A C_A^+ \frac{G_N \tilde{m}_A}{r_A^3} (v_A^j r_A^j) r_A^i,$$

$$B^\pm \equiv \pm\frac{3}{2} \pm \frac{1}{4}(\alpha_1 - 2\alpha_2)\left(1 + \frac{2-c_{14}}{2c_+ - c_{14}}\sigma\right) - \frac{1}{4}(8+\alpha_1)\left(1 + \frac{c_-}{2c_1}\sigma\right),$$

$$C^\pm \equiv \frac{8+\alpha_1}{8c_1}[c_- - (1-c_-)\sigma] \pm \frac{2-c_{14}}{2}\left(\frac{2\alpha_2 - \alpha_1}{2(c_1 + 2c_3 - c_4)} + \frac{1}{c_{123}}\sigma\right)$$

AE theory: Foster 2007;
khronometric theory: Yagi, Blas, Yunes & EB 2013

Specialize to one body: we can extract sensitivity from the asymptotic metric of a **slowly moving [i.e. O(v)] neutron star** solution

e.g. in AE theory $\sigma = \frac{2c_1(2A - 4 - \alpha_1)}{(c_1 - c_3)(8+\alpha_1)}$ $\quad A = -(B^- + B^+ + 2)$

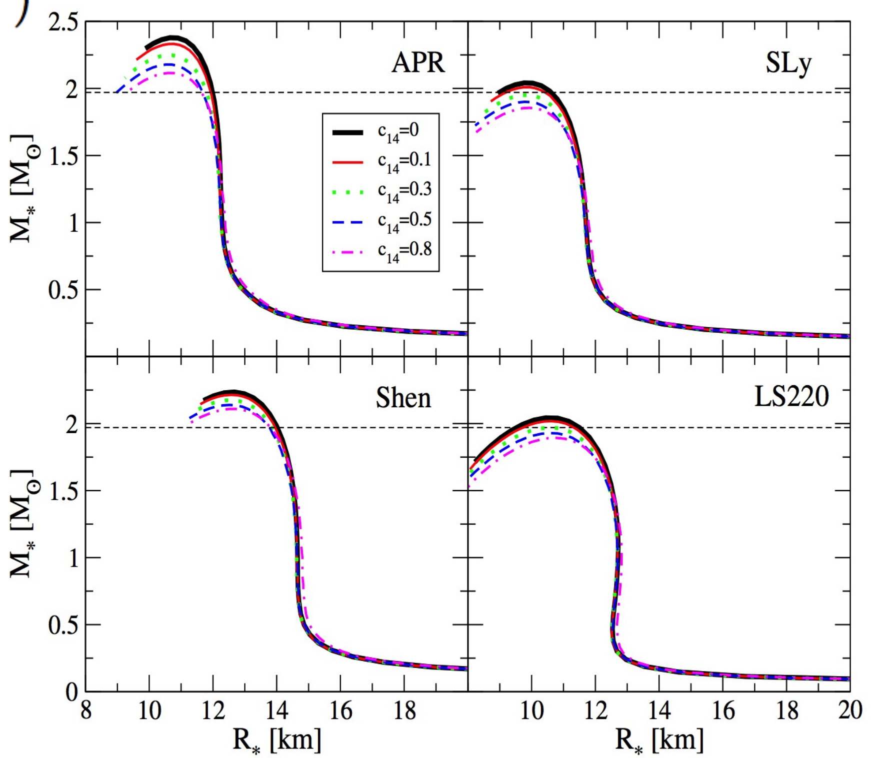
Neutron stars at order $O(v)^0$

- Static spherically symmetric, asymptotically flat solutions, same in AE and khronometric theory
- Aether and fluid are at rest

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\boldsymbol{U} = \boldsymbol{u} = e^{\nu(r)/2}dt$$

- Solutions found by imposing
  - regularity at center
  - junction condition at surface (i.e. no jumps in potentials whose derivatives enter field eqs)
  - asymptotic flatness
- Various EOS



Neutron stars at order O(v)

- Stationary and axisymmetric around velocity direction (z)
- By symmetry under *t* → - *t*, *z* → - *z*, *v* → - *v*, most generic ansatz is

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + 2vV(r,\theta)dtdr + 2vrS(r,\theta)dtd\theta + \mathcal{O}(v^2)$$

$$\boldsymbol{u} = e^{\nu(r)/2}dt + vA(r,\theta)dr + vB(r,\theta)d\theta + \mathcal{O}(v^2)$$

$$\boldsymbol{U} = e^{\nu(r)/2}dt + vW(r,\theta)dr + vQ(r,\theta)d\theta + \mathcal{O}(v^2)$$

$$\rho = \rho(r) + \mathcal{O}(v)^2 \qquad p = p(r) + \mathcal{O}(v)^2$$

- $x' = x + v\,\delta x$ sets $A = B = 0$ (comoving gauge)
- $t' = t + v\,\delta t$ sets $Q = 0$ (AE) or $Q = W = 0$ (khrono)

Boundary conditions:
- regularity at center
- asymptotically, $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}(1/r) \qquad U^\mu = (\partial_t^\mu + v\partial_z^\mu) + \mathcal{O}(1/r)$

Neutron stars at order O(v)

- System of 3 (AE) or 2 (krono) coupled PDEs in *r* and θ

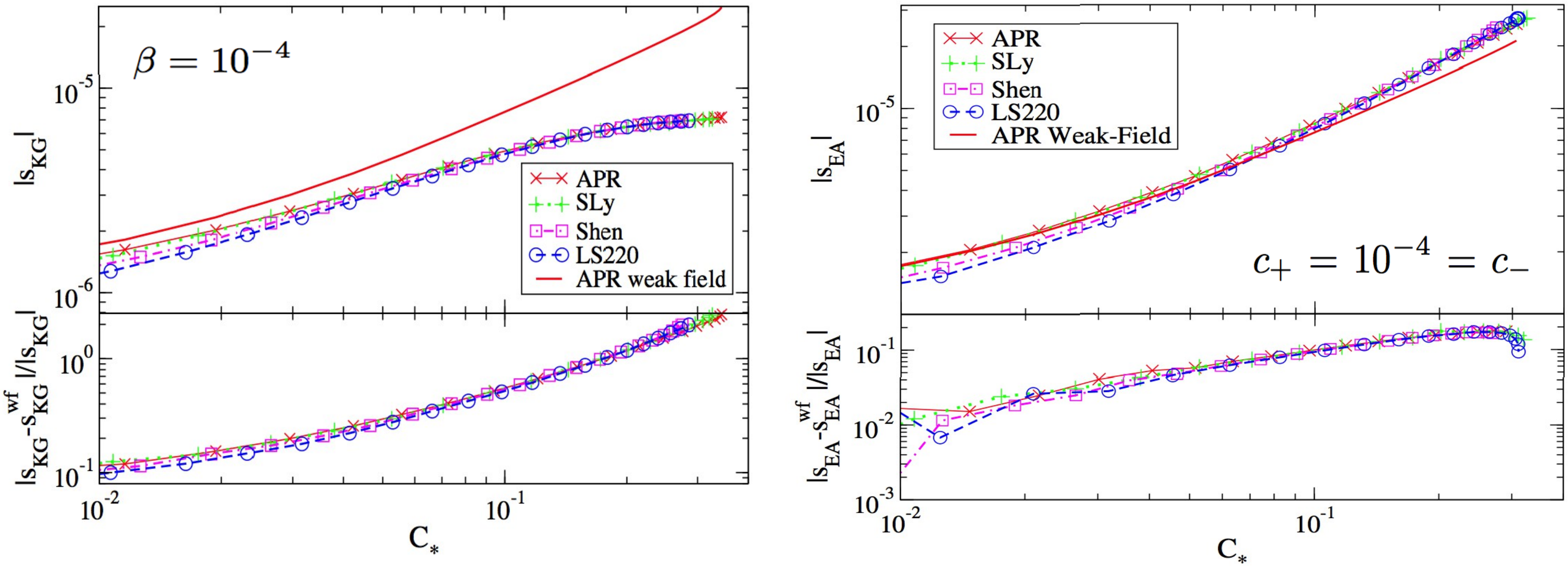
- Can be solved by expanding in Legendre polynomials

$$V(r,\theta) = \sum_n k_n(r) P_n(\cos\theta) \qquad S(r,\theta) = \sum_n s_n(r) \frac{dP_n(\cos\theta)}{d\theta} \qquad W(r,\theta) = \sum_n w_n(r) P_n(\cos\theta)$$

- Eqs for $k_n(r)$, $s_n(r)$, $w_n(r)$ decouple: system reduces to an infinite number of ODEs

- Coefficient B_+ and B_- needed to calculate sensitivity appears for $n=1$: we can just solve 3 (AE) or 2 (krono) coupled ODEs and read off asymptotic behavior

Results: the sensitivity of neutron stars



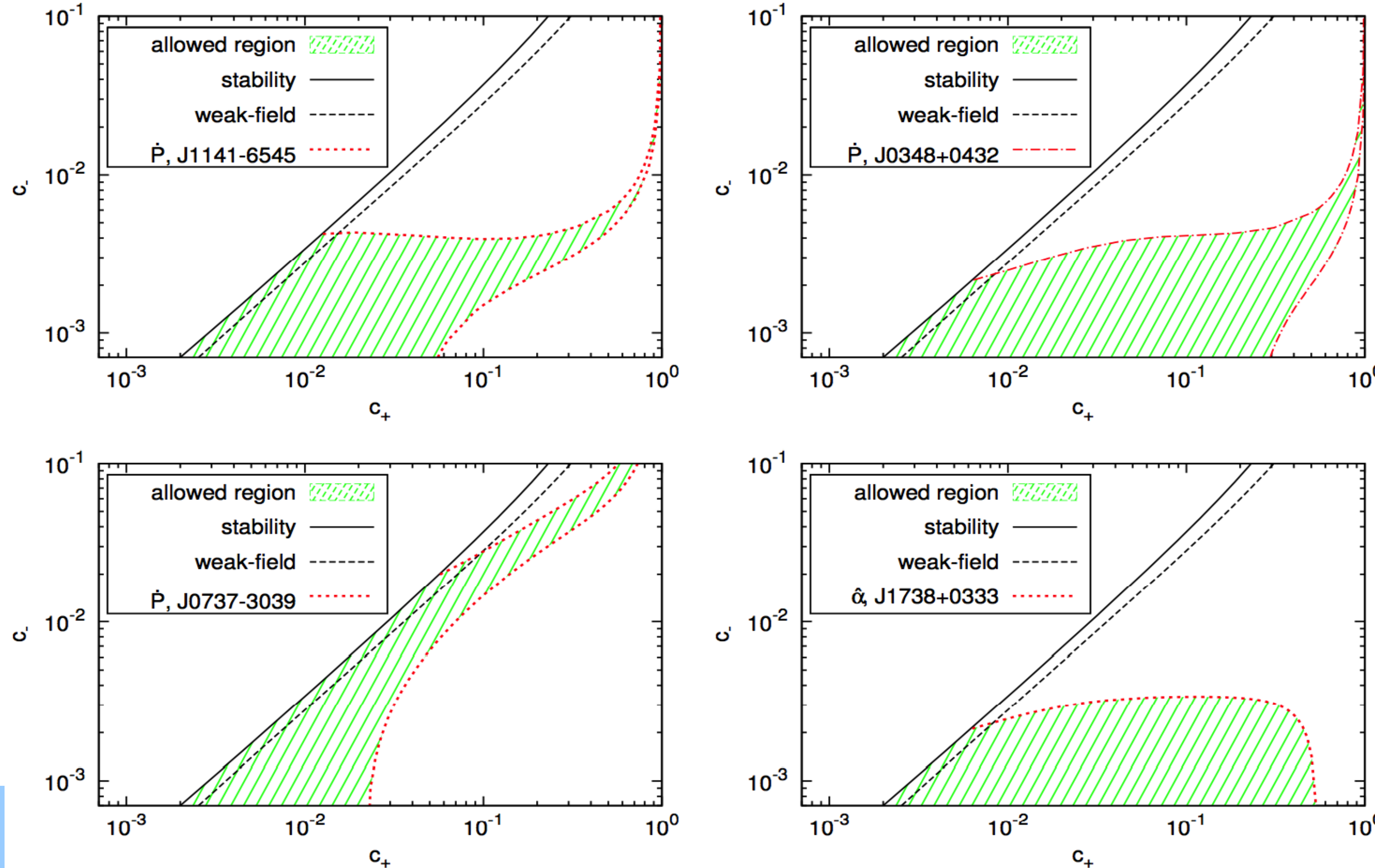
$C_* = M_* / R_*$ $\alpha_1 = 10^{-4}$ $\alpha_2 = 4 \times 10^{-7}$

Red = weak field prediction (Foster 2007)

$$s_{\rm wf} = \left(\alpha_1 - \frac{2}{3}\alpha_2\right)\frac{\Omega}{M_*} + \mathcal{O}\left(\frac{\Omega^2}{M_*^2}\right)$$

Constraints from binary pulsars

We choose pulsar-pulsar and pulsar-WD binaries with small eccentricities (PSR J1141-6545, PSR J0348+0432, PSR J0737-3039, PSR J1738+0333), and impose that difference from GR is < data uncertainties



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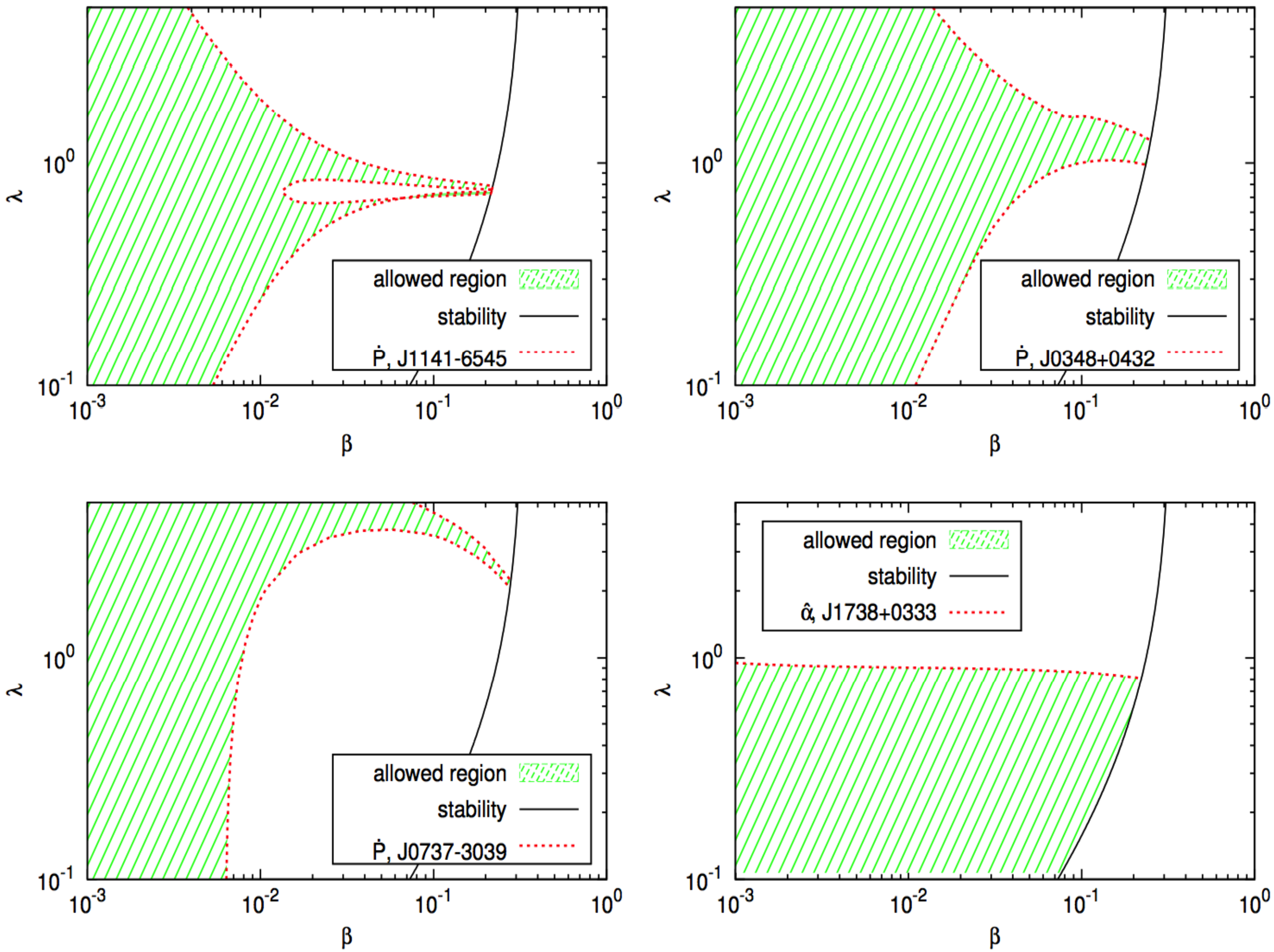
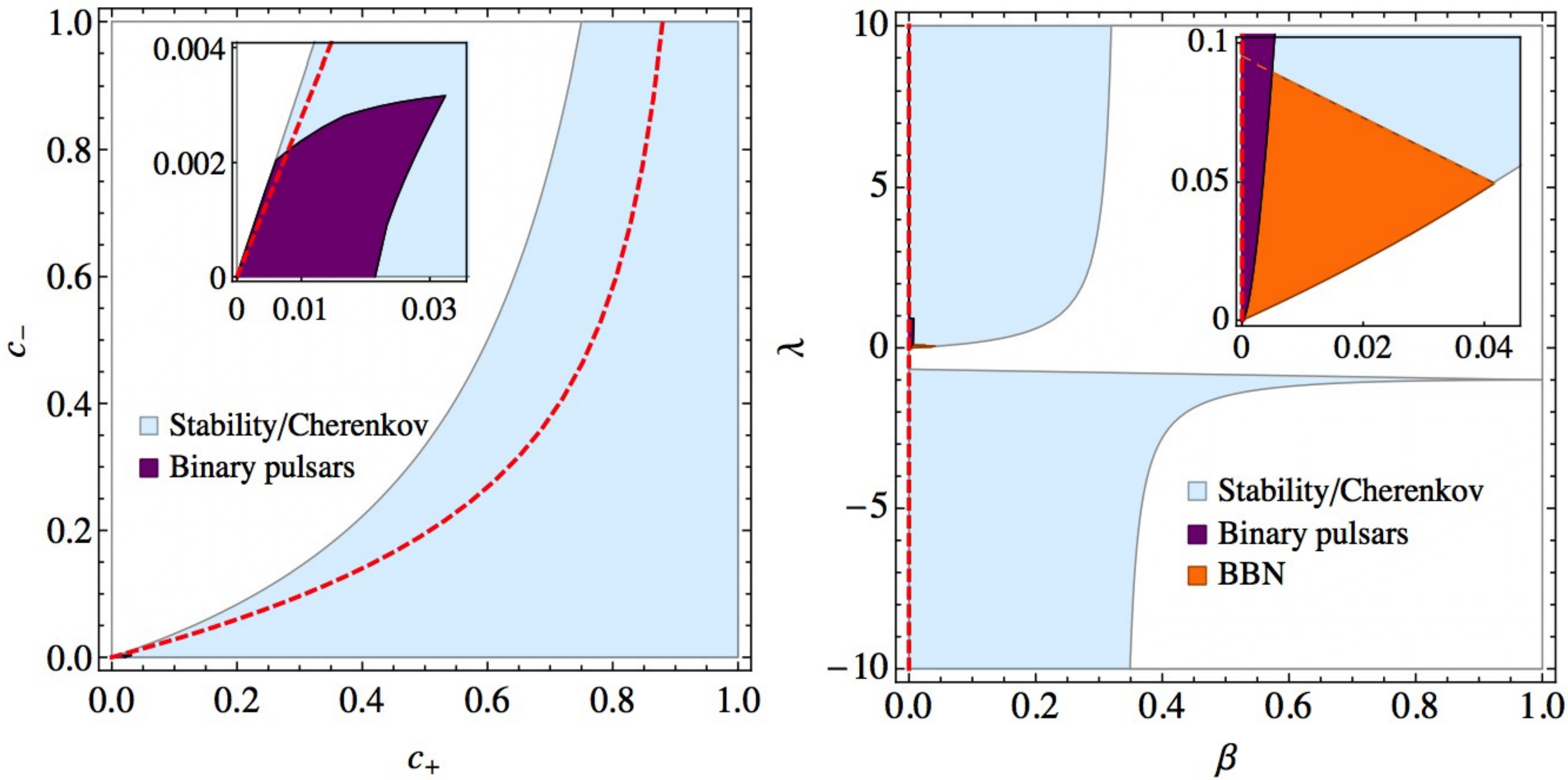


Constraints on Lorentz violation in gravity



- Red = weak field prediction for $\alpha_1 = \alpha_2 = 0$ (by requiring exactly same fluxes as GR)
- Combined constraints from WD-pulsar and pulsar-pulsar systems (PSR J1141-6545, PSR J0348+0432, PSR J0737-3039, PSR J1738+0333)
- Includes observational uncertainties (masses, spins, eccentricity, EOS)

Conclusions

- Lorentz violations in gravity generically introduces violations of strong equivalence principle and thus dipole emission
- Placing precise constraints with binary pulsars requires exact calculation of sensitivities (weak field approximation inadequate)
- Sensitivities can be obtained exactly from slowly moving, strong-field neutron star solutions
- Resulting constraints are strong-field and ~ order of magnitude stronger than previous ones

Why don't we need the binary's velocity relative to the aether?

$$\frac{\dot{P}_b}{P_b} = -\frac{192\pi}{5}\left(\frac{2\pi G_N m}{P_b}\right)^{5/3}\left(\frac{\mu}{m}\right)\frac{1}{P_b}\langle\mathcal{A}\rangle$$

$$\begin{aligned}\langle\mathcal{A}\rangle \equiv & \frac{5(1-c_{14}/2)}{32}(s_1-s_2)^2\left(\frac{P_b}{2\pi G_N m}\right)^{2/3}\mathcal{C} \\ & \times\left[1+\mathcal{O}\left(\frac{v^2}{c^2},\frac{V_{CM}^2}{c^2},(s_1-s_2)^{-1}\frac{V_{CM}v}{c^2}\right)\right] \\ & +\left(1-\frac{c_{14}}{2}\right)\left[(1-s_1)(1-s_2)\right]^{2/3} \\ & \times\left(\mathcal{A}_1+\mathcal{S}\mathcal{A}_2+\mathcal{S}^2\mathcal{A}_3\right)\left[1+\mathcal{O}\left(\frac{v^2}{c^2}\right)\right].\end{aligned}$$

$V_{cm}/c \sim V_{cmb}/c \sim 1.e\text{-}3$ is subleading

A cosmological motivation for LV gravity?

- Vector fields seem necessary to produce relativistic versions of MOND (cf. TeVeS)
- Khronometric theory with higher order terms can produce MOND-like behavior (Blanchet & Marsat 2011)

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi}\Big[R - 2f(a)\Big] + \mathcal{L}_{\rm m}[g_{\mu\nu}, \Psi]\,;$$ (covariant formulation)

$$\mathcal{L} = \frac{\sqrt{\gamma}}{16\pi}N\Big[\mathcal{R} + K_{ij}K^{ij} - K^2 - 2f(a)\Big] + \mathcal{L}_{\rm m}[N, N_i, \gamma_{ij}, \Psi]$$ (3+1 formulation)

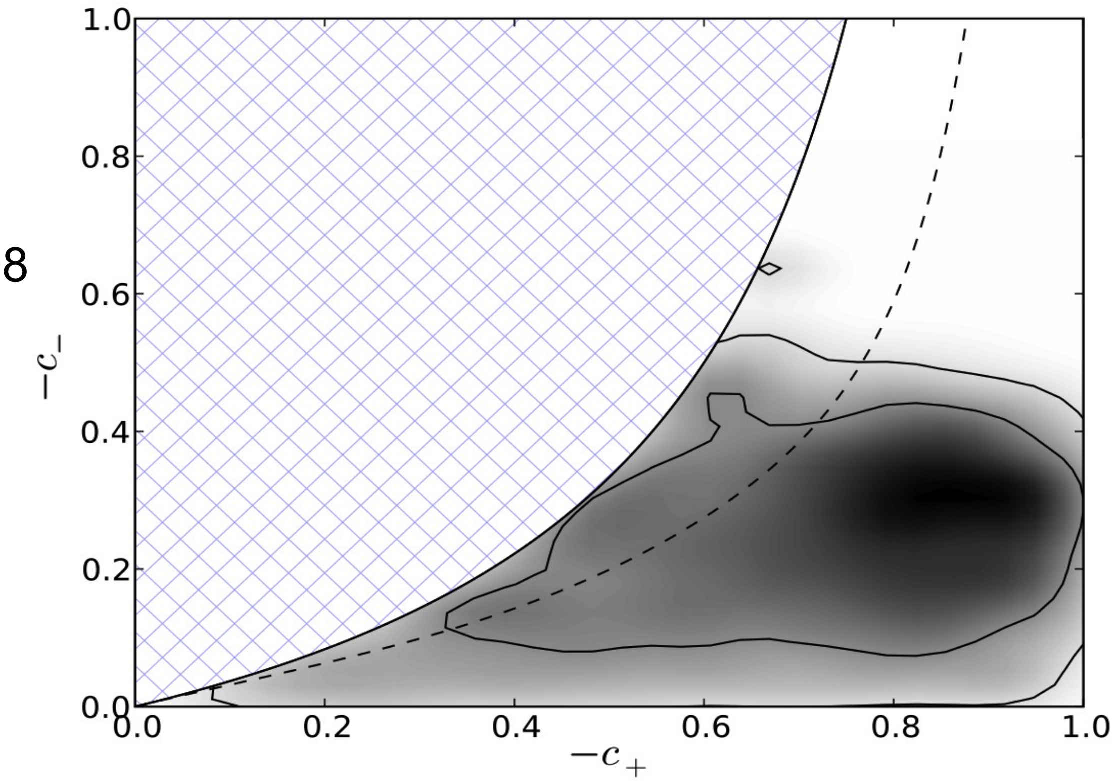
$$f(a) = \Lambda - \frac{a^3}{a_\Lambda\left[e^{a/a_\Lambda} - 1\right]} \qquad a_\Lambda = \frac{1}{2\pi}\sqrt{\frac{\Lambda}{3}}\,.$$

In the Newtonian limit, $\nabla \cdot \left(\frac{|\nabla\phi|}{a_0}\nabla\phi\right) = 4\pi G\rho$ $\qquad a_0 = \frac{4a_\Lambda c^2}{3} \simeq 1.2\times 10^{-10}\,\mathrm{m/s^2}$

How about cosmological constraints?

- Gravitational constant G_C appearing in Friedmann eqs is the same as locally measured G_N in AE theory (if $\alpha_1 = \alpha_2 = 0$) → cosmological constraints (e.g. BBN, CMB) are weak

Zuntz, Ferreira and Zlosnik 2008



The field equations

- Different in AE and khronometric theory

$$E_{\alpha\beta} \equiv G_{\alpha\beta} - T^{æ}_{\alpha\beta} - 8\pi G_{æ} T_{\alpha\beta} = 0$$

$$Æ^{\nu} \equiv \left(\nabla_\alpha J^{\alpha\nu} - c_4 \dot{U}_\alpha \nabla^\nu U^\alpha\right)(g_{\mu\nu} - U_\mu U_\nu) = 0$$

$$\begin{aligned} T^{æ}_{\alpha\beta} = & \nabla_\mu \left(J_{(\alpha}{}^{\mu} U_{\beta)} - J^{\mu}{}_{(\alpha} U_{\beta)} - J_{(\alpha\beta)} U^\mu\right) \\ & + c_1 \left[(\nabla_\mu U_\alpha)(\nabla^\mu U_\beta) - (\nabla_\alpha U_\mu)(\nabla_\beta U^\mu)\right] \\ & + \left[U_\nu(\nabla_\mu J^{\mu\nu}) - c_4 \dot{U}^2\right] U_\alpha U_\beta + c_4 \dot{U}_\alpha \dot{U}_\beta \\ & + \frac{1}{2} M^{\sigma\rho}{}_{\mu\nu} \nabla_\sigma U^\mu \nabla_\rho U^\nu g_{\alpha\beta}\,, \end{aligned}$$

AE theory

$$E_{\alpha\beta} - 2Æ_{(\alpha} u_{\beta)} = 0\,,$$

$$\nabla_\mu \left(\frac{Æ^\mu}{\sqrt{\nabla^\alpha T \nabla_\alpha T}}\right) = 0$$

T-equation implied by Einstein eqs and by matter stress-energy conservation, thanks to Bianchi identity

khronometric theory

- Hypersurface-orthogonal solutions to AE theory are solutions to khronometric theory, but not vice versa. E.g. spherical BHs are the same, but rotating BHs are different

Why do the equations decouple?

- Easy to see in isotropic cylyndric coordinates

$$ds^2 = f(r)dt^2 - h(r)(d\rho^2 + \rho^2 d\theta^2 + dz^2) \qquad U_\mu = \delta_\mu^t \sqrt{f(r)} \qquad r = \sqrt{\rho^2 + z^2}$$

- Symmetry $t \to -t$, $z \to -z$, $v \to -v$ allows perturbations to g_{tz}, $g_{t\rho}$, U_t, U_ρ, U_z, but normalization kills δU_t

- δg_{tz}, $\delta g_{t\rho}$, δU_t, δU_ρ, δU_z constructed with 2 vectors $n=(\rho,z)/r$ and $v=(0,z)$

$$\begin{pmatrix}\delta g_{t\rho}\\ \delta g_{tz}\end{pmatrix} = S(r)\boldsymbol{v} + V(r)(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$$

$$\begin{pmatrix}\delta U_\rho\\ \delta U_z\end{pmatrix} = \sqrt{f(r)}\left[Q(r)\boldsymbol{v} + W(r)(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}\right]$$

$$ds^2 = f(r)dt^2 - h(r)(d\rho^2 + \rho^2 d\theta^2 + dz^2)$$
$$+ 2v\left[S(r) + V(r)\frac{z^2}{r^2}\right] dz dt + 2vV(r)\frac{z\rho}{r^2} d\rho dt + \mathcal{O}(v)^2$$

$$U_\mu = \sqrt{f(r)}\left\{\delta_\mu^t + v\left[\tilde{Q}(\rho,z)\delta_\mu^z + \tilde{W}(\rho,z)\delta_\mu^\rho\right]\right\} + \mathcal{O}(v)^2$$

$$\tilde{Q}(\rho,z) = Q(r) + W(r)\frac{z^2}{r^2}$$

$$\tilde{W}(\rho,z) = W(r)\frac{z\rho}{r^2}$$

- Field eqs give system of ODEs for S(r), V(r), Q(r) and W(r)

A PN analysis: the violation of the SEP

Conservative dynamics at 1 PN depends on sensitivities (Foster 2007)

$$
\begin{aligned}
\dot{v}_1^i = & \frac{\mathcal{G} m_2}{r^2} \hat{r}^i \Big[-1 + 4\frac{\tilde{m}_2}{r} + \Big(1 - \frac{2}{1+\sigma_2} D\Big) \frac{\tilde{m}_1}{r} \\
& - \frac{1}{2}\Big(2 + 3\sigma_1 + \frac{\sigma_1'}{1+\sigma_1}\Big) v_1^2 - \Big(\frac{3}{2}(1+\sigma_2) + (E-D)\Big) v_2^2 \\
& - 2D v_1^j v_2^j + 3(E-D)(v_2^j \hat{r}^j)^2 \Big] \\
& + \frac{\mathcal{G} m_2}{r^2} \Big[v_1^i \Big(v_1^j \hat{r}^j \big(4 + 3\sigma_1 - \frac{\sigma_1'}{1+\sigma_1}\big) - 3(1+\sigma_1) v_2^j \hat{r}^j \Big) \\
& + v_2^i (2D v_1^j \hat{r}^j - 2E v_2^j \hat{r}^j) \Big],
\end{aligned}
$$

$$
D = -\frac{1}{4}(8+\alpha_1)\Big(1 + \frac{c_-}{2c_1}(\sigma_1+\sigma_2) + \frac{(1-c_-)}{2c_1}\sigma_1\sigma_2\Big),
$$

$$
E = -\frac{3}{2} - \frac{1}{4}(\alpha_1 - 2\alpha_2)\Big(1 + \frac{(2-c_{14})}{(c_1+2c_3-c_4)}(\sigma_1+\sigma_2) + \frac{(2-c_{14})}{2c_{123}}\sigma_1\sigma_2\Big)
$$

A strong-field derivation of the sensitivities

- Explicitly $\delta M = \frac{1}{4G}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta\, r^2 \sin\theta\, \delta U^\mu J^r{}_\mu$ $\qquad J^{\alpha\mu} = M^{\alpha\beta\mu\nu}\nabla_\beta U_\nu$
- From $U^\mu = (\partial_t^\mu + v\partial_z^\mu)/\sqrt{1-v^2} + \mathcal{O}(1/r)$

$$\delta U^\mu = \delta v \left(v\delta_t^\mu + \cos\theta\delta_r^\mu - \frac{\sin\theta}{r}\delta_\theta^\mu \right)\left[1 + \mathcal{O}\left(\frac{1}{r}, \delta v\right)\right]$$

- Solve field equations far from the system to calculate J:

$$\boldsymbol{U} = \left(1 - \frac{G_N M}{r}\right) dt + v\left(1 - (1 + B^- + B^+ + C^- + C^+)\frac{G_N M}{r}\right)\cos\theta dr - vr\left(1 - (3 + 2B^- + 2C^-)\frac{G_N M}{2r}\right)\sin\theta d\theta$$

$$ds^2 = \left(1 - \frac{2G_N M}{r}\right) dt_c^2 - \left(1 + \frac{2G_N M}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$
$$-2v(B^- + B^+ + 4)\frac{G_N M}{r}\cos\theta dt dr + v(7 + 2B^-)G_N M \sin\theta dt d\theta$$

- Finally $\delta M = -\bar{\sigma}_{Æ} M v\, \delta v\, [1 + \mathcal{O}(v, \delta v)]$ with $\bar{\sigma}_{Æ} = -\frac{2c_1\left[2\left(B^+ + B^-\right) + 8 + \alpha_1\right]}{\left(c_1 - c_3\right)\left(8 + \alpha_1\right)}$

$$\Rightarrow \frac{\partial \log M}{\partial v}(v) = -\bar{\sigma}_{Æ}\, v \quad \Rightarrow \quad \sigma = -\left.\frac{\partial^2 \log M}{\partial v^2}\right|_{v=0} = \bar{\sigma}_{Æ}$$

A strong-field derivation of the sensitivities

- Similar procedure for khronometric theory

$$\delta M = -\int d^2 S_i \left[\frac{\partial \mathcal{L}_U}{\partial(\partial_i U_\mu)}\delta U_\mu + \frac{(\delta^i_\mu - U^i U_\mu) Æ^\mu \delta T}{\sqrt{g^{\alpha\beta}\partial_\alpha T \partial_\beta T}}.\right] \qquad Æ^\mu = \frac{\partial \mathcal{L}_U}{\partial U_\mu} - \partial_i \left[\frac{\partial \mathcal{L}_U}{\partial(\partial_i U_\mu)}\right]$$

$$= -\frac{1}{8\pi G}\sqrt{-g}\left(c_4 a^\nu \nabla^\mu U_\nu - \nabla_\alpha J^{\alpha\mu}\right)$$

$$\delta M = \frac{1}{4G}\lim_{r\to\infty}\int_0^\pi \mathrm{d}\theta\, r^2 \sin\theta \quad \times \left[\delta U_\mu J^{r\mu} + \left(c_4 a^\nu \nabla^r U_\nu - \nabla_\alpha J^{\alpha r}\right)\delta T\right] + \mathcal{O}(v^2)$$

$$\frac{\partial \log M}{\partial v}(v) = -\bar{\sigma}\, v$$

$$\sigma = -\left.\frac{\partial^2 \log M}{\partial v^2}\right|_{v=0} = \bar{\sigma}_H = -1 + \frac{2\left(B^+ + B^-\right)}{8+\alpha_1}$$

To get sensitivity, we only need slowly-moving NS solution!