Strong Binary Pulsar Constraints on Lorentz Violation in Gravity

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based on
Kent Yagi, Diego Blas, Nicolas Yunes, EB, arXiv:1307.6219

Virtual Institute of Astroparticle Physics, November 8, 2013
Outline

• Lorentz violation in gravity: why and how
  Einstein aether theory and khronometric theory (= low-energy Horava gravity)

• Lorentz violation implies violations of the strong equivalence principle:
  The motion of neutron stars (the “sensitivities” and dipolar gravitational-wave emission)

• Constraints from binaries containing pulsar
Lorentz violation in gravity: why?

- LV may give better UV behavior (Horava), QG completions generally lead to LV

- LV allows MOND-like (Bekenstein, Blanchet & Marsat) or DE-like phenomenology

- Strong constraints in matter sector, weaker ones in gravity sector (caveat: constraints expected to percolate from gravity to matter sector)

- Solar system/isolated & binary pulsar experiments historically used to constrains LV in weak field (1 PN) regimes ("preferred-frame parameters": Nordvedt, Kramer, Wex, Damour, Esposito Farese), but surprises may happen in stronger-field regimes
Einstein-aether theory

• We want to specify a (local) preferred time “direction” timelike aether field $U_\mu$ with unit norm

• Most generic action (in 4D) that's quadratic in derivatives is given (up to total derivatives) by

$$S_{ae} = \frac{1}{16\pi G_{ae}} \int d^4 x \sqrt{-g} \left( -R - M^{\alpha\beta}_{\mu\nu} \nabla_\alpha U^\mu \nabla_\beta U^\nu \right)$$

$$M^{\alpha\beta}_{\mu\nu} = c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta^\alpha_\mu \delta^\beta_\nu + c_3 \delta^\alpha_\nu \delta^\beta_\mu + c_4 U^\alpha U^\beta g_{\mu\nu}$$

• To satisfy weak equivalence principle, matter fields couple minimally to metric (and not directly to aether)

$$S = S_{ae} + S_{\text{matter}}(\psi, g_{\mu\nu})$$
Khronometric gravity

- To specify a global time, $U$ must be hypersurface orthogonal ("khronometric" theory)

$$U_\mu = \frac{\partial_\mu T}{\sqrt{g^{\mu\nu} \partial_\mu T \partial_\nu T}} \quad S_\infty = \frac{1}{16\pi G_\infty} \int d^4x \sqrt{-g} \left( -R - M^{\alpha\beta}_{\mu\nu} \nabla_\alpha U^\mu \nabla_\beta U^\nu \right)$$

- Because $U$ is timelike, $T$ can be used to as time coordinate

$$U_\alpha = \delta^T_\alpha (g^{TT})^{-1/2} = N\delta^T_\alpha \quad a_i = \partial_i \ln N$$

$$S_K = \frac{1}{16\pi G_K} \int dT d^3x N\sqrt{h} \left( K_{ij}K^{ij} - \mu K^2 + \xi^{(3)}R + \eta a_i a^i \right)$$

- 3 free parameters vs 4 of AE theory (because aether is hypersurface orthogonal)
Khronometric vs Horava gravity

\[ S_H = \frac{1}{16\pi G_K} \int dT d^3x \sqrt{h} \left( L_2 + \frac{1}{M_*^2} L_4 + \frac{1}{M_*^4} L_6 \right) \]

\[ L_2 = K_{ij} K^{ij} - \mu K^2 + \xi^{(3)} R + \eta a_i a^i, \]

- \( L_4 \) and \( L_6 \) contain 4th- and 6th-order terms in the spatial derivatives.
- Lower bound on \( M_* \) depends on details of percolation of Lorentz violations from gravity to matter: from Lorentz violations in gravity alone, \( M_* \gtrsim 10^{-3} \text{ eV} \), but precise bounds depend on percolation.
- Theory remains perturbative at all scales if \( M_* \lesssim 10^{16} \text{ GeV} \).
- Terms crucial in the UV, but unimportant astrophysically, ie error scales as \( \sim \frac{M_{\text{Planck}}^4}{(MM_*)^2} \sim 10^{-14} (M_\odot/M)^2 \).
Constraints on the coupling constants: the Parametrized Post-Newtonian expansion

- At 1PN, theories = to GR except for preferred-frame parameters $\alpha_1$ and $\alpha_2$ which are zero in GR but not in LV gravity
- Solar system & pulsar experiments require $|\alpha_1| \lesssim 10^{-5}$, $|\alpha_2| \lesssim 10^{-9}$
- Imposing $\alpha_1 = \alpha_2 = 0$ reduces couplings from 4 to 2 (AE theory)
  $c_{\pm} = c_1 \pm c_3$
- ... and from 3 to 2 (khronometric theory) $\lambda = \mu/\xi - 1$, $\beta = (\xi - 1)/\xi$
- $c_+$, $c_-$ and $\lambda$, $\beta$ enter at PN order > 1 (they are “stronger-field”)
Constraints on the coupling constants: stability

- AE theory has propagating spin-0, spin-1 and spin-2 gravitational modes
- Khronometric theory has spin-0, spin-2 modes
- For stability, propagation speeds need to be real (no tachyonic instability)
- Propagation speed must be larger than speed of light to avoid gravitational Cherenkov radiation
Stability + Cherenkov constraints

\[ c_{\pm} = c_1 \pm c_3 \]

\[ \lambda = \frac{\mu}{\xi} - 1, \quad \beta = \frac{(\xi - 1)}{\xi} \]
How about cosmological constraints?

- Weak for AE theory

- For khronometric theory,

\[ \frac{G_N}{G_C} = \frac{2 + \beta + 3\lambda}{2(1 - \beta)} \]

and BBN requires

\[ |\frac{G_N}{G_C} - 1| < 1/8 \]

- No constraints from CMB in khronometric theory yet
Why are astrophysical effects expected?

- Matter couples minimally to metric, but metric couples non-minimally to aether ➡️ effective matter-aether coupling in strong-field regimes

- For strongly gravitating body (e.g. neutron star), binding energy depends on velocity relative to the aether \( \gamma = U_\mu u^\mu \)
  (i.e. structure depends on motion relative to preferred frame, as expected from Lorentz violation!)

- Gravitational mass depends on velocity relative to the aether

\[
S_{\text{matter}} = \sum \int m_i (\gamma) \, d\tau_i \quad \Rightarrow \quad u_a^\mu \nabla_\mu (m_a u^\nu) = -\frac{d}{d\gamma} m^a u^\mu \nabla^\nu U_\mu
\]

Violations of strong equivalence principle (aka Nordtvedt effect in Brans Dicke theory, scalar tensor theories, etc)
Whenever strong equivalence principle (SEP) is violated, dipolar gravitational-wave emission may be produced

- In GR, dipolar emission not present because of SEP + conservation of linear momentum

\[ M_1 \equiv \int \rho x_i \, d^3 x \quad h \sim \frac{G}{c^3} \frac{d}{dt} \frac{P}{r} \sim \frac{G}{c^3} \frac{P}{r} \text{ not a wave!} \]

- If SEP is violated,

\[ h \sim \frac{1}{R} \frac{d}{dt} [m_1(\gamma)x_1 + m_2(\gamma)x_2] \sim \frac{\mu}{R} \nu \left( \frac{d \log m_1}{d \log \gamma} - \frac{d \log m_2}{d \log \gamma} \right) \]

- Dipolar mode might be observable directly by interferometers, or indirectly via its backreaction on a binary's evolution
A PN analysis: the violation of the SEP

\[ S_A = - \int d\tau \tilde{m}_A[\gamma] = -\tilde{m}_A \int d\tau \left\{ 1 + \sigma_A (1 - \gamma_A) + \frac{1}{2} \sigma'_A (1 - \gamma_A)^2 + \mathcal{O} \left[(1 - \gamma_A)^3]\right\} \]

\[ \gamma = U^\mu u_\mu \quad \sigma_A \equiv -\left. \frac{d \ln \tilde{m}_A[\gamma_A]}{d \ln \gamma_A} \right|_{\gamma_A = 1} \quad \sigma'_A \equiv \sigma_A + \sigma_A^2 + \left. \frac{d^2 \ln \tilde{m}_A[\gamma_A]}{d \ln \gamma_A^2} \right|_{\gamma_A = 1} \text{ body's "sensitivities"} \]

Define "active" gravitational mass \( m_A = (1 + \sigma_A)\tilde{m}_A \)

and "strong-field" gravitational constant \( G_{AB} = \frac{G_N}{(1 + \sigma_A)(1 + \sigma_B)} \)

Modified Newton's law:

\[ \ddot{u}_A^i = \sum_{B \neq A} \frac{-G_N \tilde{m}_B}{(1 + \sigma_A) r_{AB}^3} r_{AB}^i \equiv \sum_{B \neq A} \frac{-G_{AB} m_B}{r_{AB}^3} r_{AB}^i \quad \text{Foster 2007} \]
A PN analysis: the dissipative dynamics

- GWs carry energy away from binaries

\[ \dot{E} = -\frac{32}{5} G_N (G_N M)^{4/3} \mu^2 \left( \frac{P_b}{2\pi} \right)^{-10/3} \langle A \rangle \]

\[ \langle A \rangle = \frac{1}{(1 + \sigma_1)^{4/3} (1 + \sigma_2)^{4/3}} \left[ A_1 + S A_2 + S^2 A_3 + \frac{5}{32} (s_1 - s_2)^2 C (1 + \sigma_1)^{2/3} (1 + \sigma_2)^{2/3} \left( \frac{P_b}{2\pi G_N M} \right)^{2/3} \right] \]

- Dipole
- Quadrupole

\[ S = (s_1 m_2 + s_2 m_1) / M \]
\[ M = m_1 + m_2 \]
\[ \mu = \frac{m_1 m_2}{M}, \quad s_A = \sigma_A / (1 + \sigma_A) \]

\[ \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \text{ are functions of the coupling constants } (c_+, c_-) \text{ or } (\beta, \lambda); \]

in GR \( \mathcal{A} = 1 \) (Foster 2007, Yagi, Blas, Yunes, EB 2013)

- As binary's binding energy decreases, period decreases

\[ \frac{\dot{P}_b}{P_b} = -\frac{3}{2} \frac{\dot{E}}{E} = \frac{3}{2} \frac{\dot{E}}{E} \]
Why is this interesting?

Binary pulsars are the strongest test of GR to date

To calculate rate of change of orbital period we need sensitivities

$$\sigma = -\left. \frac{\partial \log M}{\partial \log \gamma} \right|_{v=0} = -2 \left. \frac{\partial \log M}{\partial (v^2)} \right|_{v=0}$$

PSR B1913+16
(Weisberg & Taylor 2004)
A strong-field derivation of the sensitivities

- Consider stationary configuration describing NS and aether moving with small velocity $v$ against it, i.e. asymptotically

$$ds^2 = \left[ \left( 1 - \frac{2M_*}{r} \right) dt^2 - \left( 1 + \frac{2M_*}{r} \right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 - 2v(B^- + B^+) \frac{M_*}{r} \cos \theta dt dr \right. $$

$$\left. + v(7 + 2B^-) M_* \sin \theta dt d\theta \right] \times \left[ 1 + O \left( v, \frac{1}{r} \right) \right] \times \left[ 1 + O \left( v, \frac{1}{r} \right) \right]$$

- All fields are time-independent, so

$$U_{\mu} dx^\mu = \left\{ \left( 1 - \frac{M_*}{r} \right) dt \right. $$

$$\left. + v \left[ 1 - (1 + B^- + B^+ + C^- + C^+) \frac{M_*}{r} \right] \cos \theta dr \right\} $$

$$\left. - vr \left[ 1 - (3 + 2B^- + 2C^-) \frac{M_*}{2r} \right] \sin \theta d\theta \right\} $$

$$M = - \int_\Sigma d^3 x (\mathcal{L}_g + \mathcal{L}_U + \mathcal{L}_m)$$

$$\mathcal{L}_g = \sqrt{-g} g^{\mu \nu} \left( \Gamma^\alpha_{\mu \lambda} \Gamma^\lambda_{\nu \alpha} - \Gamma^\lambda_{\mu \nu} \Gamma^\alpha_{\lambda \alpha} \right)$$

$$\mathcal{L}_U = - \frac{1}{16\pi G} \sqrt{-g} M^\alpha_{\mu \nu} \nabla_\alpha U^\mu \nabla_\beta U^\nu$$
A strong-field derivation of the sensitivities

- Taking difference between configurations with $v$ and $v+\delta v$, bulk terms disappear (because we are on shell).
- Surface terms from metric vanish because of boundary conditions, and those from matter vanish because no matter at spatial infinity
  \[
  \delta M = - \int_{\partial \Sigma} d^2S_i \delta U^\mu \left( \frac{\partial L_U}{\partial (\partial_i U^\mu)} \right)
  \]
- Surface integral can be evaluate asymptotically to get
  \[
  \frac{\partial \log M}{\partial v}(v) = -\bar{\sigma}_E \ v
  \]
  \[
  \bar{\sigma}_E = - \frac{2c_1 [2 (B^+ + B^-) + 8 + \alpha_1]}{(c_1 - c_3) (8 + \alpha_1)}
  \]

  \[
  \sigma = - \left. \frac{\partial^2 \log M}{\partial v^2} \right|_{v=0} = \bar{\sigma}_E
  \]

  To get sensitivity, we only need slowly-moving NS solution!
The PN way to the sensitivities

Solving the field equations in the standard PPN gauge for a system of bodies with velocities $v^A$ relative to the aether (which is asymptotically at rest):

$$g_{00} = -1 + 2 \sum_A \frac{G_N m_A}{r_A} - 2 \sum_{A,B} \frac{G_N^2 m_A m_B}{r_A r_B} - 2 \sum_{A,B \neq A} \frac{G_N^2 m_A m_B}{r_A r_{AB}} + 3 \sum_A \frac{G_N m_A}{r_A} v_A^2 (1 + \sigma_A),$$

$$g^{0i} = \left(1 + 2 \sum_A \frac{G_N m_A}{r_A}\right) \delta^{0i},$$

$$g^{0i} = \sum_A B^A_{-} \frac{G_N m_A}{r_A} v_A^i + \sum_A B^A_{+} \frac{G_N m_A}{r_A} (v_A^j r_A^j) r_A^i,$$

AE theory: Foster 2007;
khronometric theory: Yagi, Blas, Yunes & EB 2013

$$u^0 = 1 + \sum_A \frac{G_N m_A}{r_A},$$

$$u^i = \sum_A C_A^{-} \frac{G_N m_A}{r_A} (v_A)^i + \sum_A C_A^{+} \frac{G_N m_A}{r_A^3} (v_A^j r_A^j) r_A^i,$$

$$B^\pm \equiv \pm \frac{3}{2} \pm \frac{1}{4} (\alpha_1 - 2 \alpha_2) \left(1 + \frac{2 - c_{14}}{2 c_+ - c_{14}} \right) \left(1 + \frac{c_-}{2 c_1} \right),$$

$$C^\pm \equiv \frac{8 + \alpha_1}{8 c_1} [c_- - (1 - c_-) \sigma] + \pm \frac{2 - c_{14}}{2} \left(\frac{2 \alpha_2 - \alpha_1}{2(c_1 + 2 c_3 - c_4)} + \frac{1}{c_{123}} \sigma\right)$$

Specialize to one body: we can extract sensitivity from the asymptotic metric of a slowly moving [i.e. $O(v)$] neutron star solution

$\sigma = \frac{2 c_1 (2 A - 4 - \alpha_1)}{(c_1 - c_3)(8 + \alpha_1)}$

$A = -(B^- + B^+ + 2)$
Neutron stars at order $O(\nu)^0$

- Static spherically symmetric, asymptotically flat solutions, same in AE and khronometric theory
- Aether and fluid are at rest
  \[ ds^2 = e^{\nu(r)} dt^2 - e^{\mu(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]
  \[ U = u = e^{\nu(r)/2} dt \]
- Solutions found by imposing
  - regularity at center
  - junction condition at surface
    (i.e. no jumps in potentials whose derivatives enter field eqs)
  - asymptotic flatness
- Various EOS
Neutron stars at order $O(v)$

- Stationary and axisymmetric around velocity direction ($z$)
- By symmetry under $t \to -t$, $z \to -z$, $v \to -v$, most generic ansatz is

$$ds^2 = e^\nu(r) dt^2 - e^\mu(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$u = e^\nu(r)/2 dt + vA(r, \theta)dr + vB(r, \theta)d\theta + O(v^2)$$

$$U = e^\nu(r)/2 dt + vW(r, \theta)dr + vQ(r, \theta)d\theta + O(v^2)$$

$$\rho = \rho(r) + O(v^2) \quad p = p(r) + O(v^2)$$

- $x' = x + v \delta x$ sets $A = B = 0$ (comoving gauge)
- $t' = t + v \delta t$ sets $Q = 0$ (AE) or $Q = W = 0$ (khrono)

Boundary conditions:
- regularity at center
- asymptotically, $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r)$
  $$U^\mu = (\partial^\mu_t + v\partial^\mu_z) + O(1/r)$$
Neutron stars at order $O(\nu)$

- System of 3 (AE) or 2 (krono) coupled PDEs in $r$ and $\theta$

- Can be solved by expanding in Legendre polynomials

\[ V(r, \theta) = \sum_n k_n(r) P_n(\cos \theta) \quad S(r, \theta) = \sum_n s_n(r) \frac{dP_n(\cos \theta)}{d\theta} \quad W(r, \theta) = \sum_n w_n(r) P_n(\cos \theta) \]

- Eqs for $k_n(r)$, $s_n(r)$, $w_n(r)$ decouple: system reduces to an infinite number of ODEs

- Coefficient $B_+$ and $B_-$ needed to calculate sensitivity appears for $n=1$: we can just solve 3 (AE) or 2 (krono) coupled ODEs and read off asymptotic behavior
Results: the sensitivity of neutron stars

\[ \beta = 10^{-4} \]

\[ C_* = \frac{M_*}{R_*} \quad \alpha_1 = 10^{-4} \quad \alpha_2 = 4 \times 10^{-7} \]

Red = weak field prediction (Foster 2007)

\[ s_{\text{wf}} = \left( \alpha_1 - 2 \frac{\alpha_2}{3} \right) \frac{\Omega}{M_*} + \mathcal{O} \left( \frac{\Omega^2}{M_*^2} \right) \]
Constraints from binary pulsars

We choose pulsar-pulsar and pulsar-WD binaries with small eccentricities (PSR J1141-6545, PSR J0348+0432, PSR J0737-3039, PSR J1738+0333), and impose that difference from GR is < data uncertainties.
Constraints from binary pulsars
Constraints on Lorentz violation in gravity

- Red = weak field prediction for $\alpha_1 = \alpha_2 = 0$ (by requiring exactly same fluxes as GR)
- Combined constraints from WD-pulsar and pulsar-pulsar systems (PSR J1141-6545, PSR J0348+0432, PSR J0737-3039, PSR J1738+0333)
- Includes observational uncertainties (masses, spins, eccentricity, EOS)
Conclusions

- Lorentz violations in gravity generically introduces violations of strong equivalence principle and thus dipole emission
- Placing precise constraints with binary pulsars requires exact calculation of sensitivities (weak field approximation inadequate)
- Sensitivities can be obtained exactly from slowly moving, strong-field neutron star solutions
- Resulting constraints are strong-field and ~ order of magnitude stronger than previous ones
Why don't we need the binary's velocity relative to the aether?

\[
\frac{\dot{P}_b}{P_b} = -\frac{192\pi}{5} \left( \frac{2\pi G_N m}{P_b} \right)^{5/3} \left( \frac{\mu}{m} \right) \frac{1}{P_b} \langle A \rangle
\]

\[
\langle A \rangle \equiv \frac{5(1 - \frac{c_{14}}{2})}{32} (s_1 - s_2)^2 \left( \frac{P_b}{2\pi G_N m} \right)^{2/3} c
\]

\[
\times \left[ 1 + \mathcal{O} \left( \frac{v^2}{c^2}, \frac{V_{CM}^2}{c^2}, (s_1 - s_2)^{-1} \frac{V_{CM} v}{c^2} \right) \right]
\]

\[
+ \left( 1 - \frac{c_{14}}{2} \right) \left[ (1 - s_1)(1 - s_2) \right]^{2/3}
\]

\[
\times (A_1 + \mathcal{S}_A_2 + S^2 \mathcal{A}_3) \left[ 1 + \mathcal{O} \left( \frac{v^2}{c^2} \right) \right].
\]

\[
V_{cm}/c \sim V_{cmb}/c \sim 1.e-3 \text{ is subleading}
\]
A cosmological motivation for LV gravity?

- Vector fields seem necessary to produce relativistic versions of MOND (cf. TeVeS)

- Khronometric theory with higher order terms can produce MOND-like behavior (Blanchet & Marsat 2011)

\[
\mathcal{L} = \frac{\sqrt{-g}}{16\pi} \left[ R - 2f(a) \right] + \mathcal{L}_m[g_{\mu\nu}, \Psi], \quad \text{(covariant formulation)}
\]

\[
\mathcal{L} = \frac{\sqrt{\gamma}}{16\pi} N \left[ \mathcal{R} + K_{ij}K^{ij} - K^2 - 2f(a) \right] + \mathcal{L}_m[N, N_i, \gamma_{ij}, \Psi] \quad \text{(3+1 formulation)}
\]

\[
f(a) = \Lambda - \frac{a^3}{a_\Lambda \left[e^{a/a_\Lambda} - 1\right]} \quad a_\Lambda = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}}.
\]

In the Newtonian limit, \[ \nabla \cdot \left( \frac{\left| \nabla \phi \right|}{a_0} \nabla \phi \right) = 4\pi G \rho \]

\[ a_0 = \frac{4a_\Lambda c^2}{3} \approx 1.2 \times 10^{-10} \text{ m/s}^2 \]
How about cosmological constraints?

- Gravitational constant $G_C$ appearing in Friedmann eqs is the same as locally measured $G_N$ in AE theory (if $\alpha_1 = \alpha_2 = 0$) cosmological constraints (e.g. BBN, CMB) are weak.
The field equations

- Different in AE and khronometric theory

\[ E_{\alpha\beta} \equiv G_{\alpha\beta} - T^{ae}_{\alpha\beta} - 8\pi G_\alpha T_{\alpha\beta} = 0 \]
\[ \dot{\varepsilon}^\nu \equiv \left( \nabla_\alpha J^{\alpha\nu} - c_4 \dot{U}_\alpha \nabla^\nu U^\alpha \right) (g_{\mu\nu} - U_\mu U_\nu) = 0 \]
\[ T^{ae}_{\alpha\beta} = \nabla_\mu \left( J^{(\mu}_{(\alpha} U_{\beta)} - J^{\mu}_{(\alpha} U_{\beta)} - J_{(\alpha\beta)} U^\mu \right) \]
\[ + c_1 \left[ (\nabla_\mu U_\alpha) (\nabla^\mu U_\beta) - (\nabla_\alpha U_\mu) (\nabla_\beta U^\mu) \right] \]
\[ + \left[ U_\nu (\nabla_\mu J^{\mu\nu}) - c_4 \dot{U}^2 \right] U_\alpha U_\beta + c_4 \dot{U}_\alpha \dot{U}_\beta \]
\[ + \frac{1}{2} M^{\rho\nu}_{\mu\nu} \nabla_\sigma U^\rho U_\nu \nabla^{\alpha\beta} g_{\alpha\beta} , \]

AE theory

- Hypersurface-orthogonal solutions to AE theory are solutions to khronometric theory, but not vice versa. E.g. spherical BHs are the same, but rotating BHs are different

\[ E_{\alpha\beta} - 2\varepsilon_{(\alpha} u_{\beta)} = 0 , \]
\[ \nabla_\mu \left( \frac{\varepsilon^\mu}{\sqrt{\nabla_\alpha T \nabla^\alpha T}} \right) = 0 \]

T-equation implied by Einstein eqs and by matter stress-energy conservation, thanks to Bianchi identity

khronometric theory
Why do the equations decouple?

• Easy to see in isotropic cylyndric coordinates

\[ ds^2 = f(r)dt^2 - h(r)(d\rho^2 + \rho^2d\theta^2 + dz^2) \quad U_\mu = \delta_\mu^t \sqrt{f(r)} \quad r = \sqrt{\rho^2 + z^2}. \]

• Symmetry \( t \to -t, \; z \to -z, \; \nu \to -\nu \) allows perturbations to \( g_{tz}, \; g_{t\rho}, \; U_t, \; U_{\rho}, \; U_z \), but normalization kills \( \delta U_t \)

• \( \delta g_{tz}, \; \delta g_{t\rho}, \; \delta U_t, \; \delta U_{\rho}, \; \delta U_z \) constructed with 2 vectors \( n=(\rho,z)/r \) and \( \nu=(0,z) \)

\[
\begin{pmatrix}
\delta g_{t\rho} \\
\delta g_{tz}
\end{pmatrix} = S(r)v + V(r)(v \cdot n)n
\]

\[
\begin{pmatrix}
\delta U_{\rho} \\
\delta U_z
\end{pmatrix} = \sqrt{f(r)} [Q(r)v + W(r)(v \cdot n)n]
\]

\[
ds^2 = f(r)dt^2 - h(r)(d\rho^2 + \rho^2d\theta^2 + dz^2) + 2v \left[ S(r) + V(r)\frac{z^2}{r^2} \right] dzdt + 2vV(r)\frac{z\rho}{r^2}d\rho dt + \mathcal{O}(v)^2
\]

\[
U_\mu = \sqrt{f(r)} \left\{ \delta_\mu^t + v \left[ \tilde{Q}(\rho,z)\delta_\mu^z + \tilde{W}(\rho,z)\delta_\mu^\rho \right] \right\} + \mathcal{O}(v)^2
\]

\[
\tilde{Q}(\rho,z) = Q(r) + W(r)\frac{z^2}{r^2}
\]

\[
\tilde{W}(\rho,z) = W(r)\frac{z\rho}{r^2}
\]

• Field eqs give system of ODEs for \( S(r), \; V(r), \; Q(r) \) and \( W(r) \)
A PN analysis: the violation of the SEP

Conservative dynamics at 1 PN depends on sensitivities (Foster 2007)

\[
\begin{align*}
\dot{v}^i_1 &= \frac{G m_2}{r^2} \ddot{r}^i \left[-1 + 4 \frac{\ddot{m}_2}{r} + \left(1 - \frac{2}{1 + \sigma_2} D \right) \frac{\ddot{m}_1}{r} \right. \\
&\quad \left. - \frac{1}{2} \left(2 + 3 \sigma_1 + \frac{\sigma'_1}{1 + \sigma_1} \right) v^2_1 - \left(\frac{3}{2} (1 - \sigma_2) + (E - D) \right) v^2_2 \right] \\
&\quad - 2 D v^j_1 v^j_2 + 3 (E - D) (v^j_2 \ddot{r}^j)^2 \\
&\quad + \frac{G m_2}{r^2} \left[ v^i_1 \left( v^j_1 \ddot{r}^j \frac{3 \sigma_1}{1 + \sigma_1} - \sigma'_1 \right) - 3 (1 + \sigma_1) v^j_2 \ddot{r}^j \right] \\
&\quad + v^i_2 \left(2 D v^j_1 \ddot{r}^j - 2 E v^j_2 \ddot{r}^j \right),
\end{align*}
\]

\[
D = -\frac{1}{4} (8 + \alpha_1) \left(1 + \frac{c_-}{2 c_1} (\sigma_1 + \sigma_2) + \frac{1 - c_-}{2 c_1} \sigma_1 \sigma_2 \right),
\]

\[
E = -\frac{3}{2} - \frac{1}{4} (\alpha_1 - 2 \alpha_2) \left(1 + \frac{2 - c_{14}}{(c_1 + 2 c_3 - c_4)} (\sigma_1 + \sigma_2) + \frac{2 - c_{14}}{2 c_{123}} \sigma_1 \sigma_2 \right).
\]
A strong-field derivation of the sensitivities

- Explicitly \[ \delta M = \frac{1}{4G} \lim_{r \to \infty} \int_0^\pi d\theta \, r^2 \sin \theta \, \delta U^\mu J_{\mu}^r \quad J^{\alpha \mu} = M^{\alpha \beta \mu \nu} \nabla_\beta U_\nu \]

- From \[ U^\mu = (\partial_t^\mu + v \partial_z^\mu)/\sqrt{1 - v^2} + \mathcal{O}(1/r) \]
  \[ \delta U^\mu = \delta v \left( v \partial_t^\mu + \cos \theta \delta_r^\mu - \frac{\sin \theta}{r} \delta_\theta^\mu \right) \left[ 1 + \mathcal{O}\left( \frac{1}{r}, \delta v \right) \right] \]

- Solve field equations far from the system to calculate \( J \):
  \[ U = \left(1 - \frac{G_N M}{r} \right) dt + v \left(1 - (1 + B^- + B^+ + C^- + C^+) \frac{G_N M}{r} \right) \cos \theta dr - vr \left(1 - (3 + 2B^- + 2C^-) \frac{G_N M}{2r} \right) \sin \theta d\theta \]
  \[ ds^2 = \left(1 - \frac{2G_N M}{r} \right) dt^2 - \left(1 + \frac{2G_N M}{r} \right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \]
  \[ -2v(B^- + B^+ + 4) \frac{G_N M}{r} \cos \theta dt dr + v(7 + 2B^-)G_N M \sin \theta dt d\theta \]

- Finally \[ \delta M = -\bar{\sigma}_AE M v \, \delta v \left[ 1 + \mathcal{O}(v, \delta v) \right] \text{ with } \bar{\sigma}_AE = -\frac{2c_1 \left[ 2 (B^+ + B^-) + 8 + \alpha_1 \right]}{(c_1 - c_3) (8 + \alpha_1)} \]

\[ \frac{\partial \log M}{\partial v} (v) = -\bar{\sigma}_AE \, v \quad \sigma = -\left. \frac{\partial^2 \log M}{\partial v^2} \right|_{v=0} = \bar{\sigma}_AE \]
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- Similar procedure for khronometric theory

\[ \delta M = - \int d^2 S \left[ \frac{\partial L_U}{\partial (\partial_i U_\mu)} \delta U_\mu + \frac{(\delta_\mu - U^i U_\mu) \delta T}{\sqrt{g^{\alpha\beta} \partial_\alpha T \partial_\beta T}} \right] \]

\[ A^\mu = \frac{\partial L_U}{\partial U_\mu} - \partial_i \left[ \frac{\partial L_U}{\partial (\partial_i U_\mu)} \right] \]

\[ = - \frac{1}{8\pi G} \sqrt{-g} \left( c_4 a^\nu \nabla^\mu U_\nu - \nabla_\alpha J_\alpha^\mu \right) \]

\[ \delta M = \frac{1}{4G} \lim_{r \to \infty} \int_0^\pi d\theta r^2 \sin \theta \times \left[ \delta U_\mu J^\mu + (c_4 a^\nu \nabla^r U_\nu - \nabla_\alpha J_\alpha^\mu) \delta T \right] + O(v^2) \]

\[ \frac{\partial \log M}{\partial v}(v) = -\bar{\sigma} \ v \]

\[ \sigma = - \left. \frac{\partial^2 \log M}{\partial v^2} \right|_{v=0} = \bar{\sigma}_H = -1 + \frac{2(B^+ + B^-)}{8 + \alpha_1} \]

To get sensitivity, we only need slowly-moving NS solution!