Backreaction for Einstein-Rosen waves coupled to a massless scalar field

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Motivation

The Nobel Prize 2011 (Perlmutter, Schmidt and Riess, 1998)

The discovery of the accelerating expansion of the Universe through observations of distant supernovae.

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The fitting problem in cosmology (Ellis, Stoeger, 1987)

- The effect of inhomogeneities
  - on light propagation: exists, but it is small,
  - on the large scale structure of spacetime:
    - exists, it is small and cannot mimic cosmological constant (Green-Wald formalism),
    - exists, it is large and can mimic cosmological constant (Buchert formalism).

Motivation

- A controversy
  - Physical Review D 83:084020 (2011),
    S. Green and R. Wald.
  - Classical and Quantum Gravity 32 (2015) 215021,
    T. Buchert, M. Carfora, G.F.R. Ellis, E.W. Kolb, M.A.H. MacCallum, J.J. Ostrowski, S. Räsänen, B.F. Roukema, L. Andersson, A.A. Coley, D.L. Wiltshire.

Averaging in general relativity

- The standard approach

$$A(g_U) = g^{(0)} \ ,$$

A — the averaging operator,
g_U — the inhomogeneous metric,
$g^{(0)}$ — the effective (averaged) metric.

Unfortunately, in cosmology, we do not know g_U.

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- The alternative point of view, $g_U = g^{(0)} + h$

$$G(g_U) = 8\pi T \ ,$$

$g^{(0)}$ — the effective metric,
h — a not necessarily small contribution (represents inhomogeneities).

In cosmology, we usually assume $g^{(0)} = g_{FRW}$. Hence

$$G(g_{FRW}) = 8\pi(T^{(0)} + t^{(0)}) \ ,$$

$T^{(0)} =< T >$ — the 'averaged' energy-momentum tensor,
$t^{(0)}$ — the effective energy-momentum tensor which represents the effect of inhomogeneities (backreaction)

Averaging in general relativity

Questions

Is it possible that $t^{(0)} \sim \Lambda g_{FRW}$?

What kind of $t^{(0)}$ will one obtain assuming g_U, g_{FRW} are known?

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- An example by Charach and Malin, 1979.

$$G(g_U) = 8\pi T,$$

g_U and T contain oscillations,
$T = \alpha \tilde{T}$ — a massless scalar field, α controls the amplitude,
$g_U = g^{(0)} + h$, where only h oscillates.

The result: $G(g^{(0)}) = 8\pi(T^{SW} + T^{GW})$,

T^{SW} and T^{GW} have the null fluid form (radiation) and $T^{SW} = \alpha \tilde{T}^{SW}$.
Moreover, $T^{SW} =< T >$.

Green–Wald framework

Within this framework properties of $t^{(0)}$ may be studied in a general setting.

Direct motivation for our studies

The Green–Wald framework assumes existence of one-parameter families of solutions to Einstein equations that satify some mathematical conditions. An exact form of those solutions is not needed to predict properties of $t^{(0)}$, but examples of such families may shed some light on possible restrictions of this method and are necessary to show that the theorems are not empty.

The G-W assumptions

(i)

There exists a one-parameter family $g_{ab}(\lambda)$, $\lambda > 0$, satisfying

$$G_{ab}(g(\lambda)) = 8\pi\, T_{ab}(\lambda)\,, \quad \lambda > 0\,,$$

where $T_{ab}(\lambda)t^a(\lambda)t^b(\lambda) \geq 0$ for all timelike $t^a(\lambda)$.

(ii)

There exists a smooth background metric

$$g^{(0)}_{ab} := \lim_{\lambda \to 0} g_{ab}(\lambda)$$

$$h_{ab}(\lambda) := g_{ab}(\lambda) - g^{(0)}_{ab} \to 0 \quad \text{for} \quad \lambda \to 0\,.$$

The G-W assumptions

(iii)

Derivatives $\nabla_a h_{bc}(\lambda)$ are *bounded* in the limit $\lambda \to 0$ (do not necessarily go to zero).
No assumptions about second derivatives – can be unbounded.

(iv)

There exists a smooth tensor field

$$\mu_{abcdef} := \underset{\lambda \to 0}{\text{w-lim}} \left[\nabla_a h_{cd}(\lambda) \nabla_b h_{ef}(\lambda) \right] .$$

The weak limit

We say that a one-parameter family of smooth tensor fields $A_{a_1 \ldots a_n}(\lambda)$, $\lambda > 0$ converges *weakly* to $B_{a_1 \ldots a_n}$ as $\lambda \to 0$,

$$B_{a_1 \ldots a_n} = \underset{\lambda \to 0}{\text{w-lim}}\, A_{a_1 \ldots a_n}(\lambda)\,,$$

if for all smooth $f^{a_1 \cdots a_n}$ of compact support, we have

$$\lim_{\lambda \to 0} \int f^{a_1 \ldots a_n} A_{a_1 \ldots a_n}(\lambda) = \int f^{a_1 \ldots a_n} B_{a_1 \ldots a_n}\,.$$

The equation satisfied by the background metric

$$G_{ab}(g^{(0)}) = 8\pi\, T^{(0)}_{ab} + 8\pi\, t^{(0)}_{ab}\,.$$

where

- $T^{(0)}_{ab} := \text{w-lim}_{\lambda\to 0}\,(T_{ab}(\lambda))$ – the 'averaged matter energy-momentum tensor'
- $t^{(0)}_{ab}$ – the 'effective gravitational energy-momentum tensor', contribution from nonlinear terms in the Einstein tensor

The effective energy-momentum tensor

$$8\pi\, t^{(0)}_{ab} = \frac{1}{8}\left(-\mu^{c}{}_{c}{}^{de}{}_{de} - \mu^{c}{}_{c}{}^{d}{}_{d}{}^{e}{}_{e} + 2\,\mu^{cd}{}_{c}{}^{e}{}_{de}\right) g^{(0)}_{ab}$$
$$+ \frac{1}{2}\,\mu^{cd}{}_{acbd} - \frac{1}{2}\,\mu^{c}{}_{ca}{}^{d}{}_{bd} + \frac{1}{4}\,\mu_{ab}{}^{cd}{}_{cd}$$
$$- \frac{1}{2}\,\mu^{c}{}_{(ab)c}{}^{d}{}_{d} + \frac{3}{4}\,\mu^{c}{}_{cab}{}^{d}{}_{d} - \frac{1}{2}\,\mu^{cd}{}_{abcd}$$

Theorem (Green and Wald, 2011)

$t^{(0)}_{ab}$ is traceless and satisfies the weak energy condition.

The Einstein-Rosen metric

The line element has the form (Einstein, Rosen 1937; Rosen 1954)

$$g = \mathsf{e}^{2(\gamma-\psi)} \left(-\mathrm{d}t^2 + \mathrm{d}\rho^2\right) + \rho^2 \, \mathsf{e}^{-2\psi} \, \mathrm{d}\varphi^2 + \mathsf{e}^{2\psi} \, \mathrm{d}z^2 \, ,$$

where (cylindrical symmetry)

$$\begin{aligned} &\rho > 0, \quad -\infty < t, z < \infty, \quad 0 \leq \varphi < 2\pi, \\ &\psi = \psi(t,\rho), \qquad \gamma = \gamma(t,\rho) \, . \end{aligned}$$

Include massless minimally coupled scalar field ϕ to obtain nonvacuum solutions.

The Einstein-Rosen-scalar field solution

Field equations reduce to

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0,$$
$$\gamma' = \rho\left(\dot{\phi}^2 + \phi'^2 + \dot{\psi}^2 + \psi'^2\right),$$
$$\dot{\gamma} = 2\rho\left(\dot{\phi}\phi' + \dot{\psi}\psi'\right),$$
$$\phi'' + \frac{1}{\rho}\phi' - \ddot{\phi} = 0.$$

The energy density of the scalar field as measured by observers comoving with the coordinate system (with four-velocity $u = e^{\psi-\gamma}\partial_t$)

$$\epsilon = T_{ab}u^a u^b = \frac{1}{8\pi}\,\mathrm{e}^{2(\psi-\gamma)}\left(\dot{\phi}^2 + \phi'^2\right).$$

One-parameter family of solutions

We choose the following particular solutions of the field equations:

$$\phi_\lambda(t,\rho) = \alpha\sqrt{\lambda}\, F_\lambda(t,\rho)\,, \quad \psi_\lambda(t,\rho) = \beta\sqrt{\lambda}\, F_\lambda(t,\rho)\,, \quad \lambda > 0\,,$$

where: $F_\lambda(t,\rho) = J_0\left(\frac{\rho}{\lambda}\right)\sin\left(\frac{t}{\lambda}\right)$; λ – parameter; J_0 – Bessel function of the first kind and zero order; constants α, β – real and independent of λ.
Integrating the remaining field equations we get

$$\gamma_\lambda(t,\rho) = \frac{(\alpha^2+\beta^2)}{2\lambda}\rho^2\left[J_0^2(\frac{\rho}{\lambda}) + J_1^2(\frac{\rho}{\lambda}) - 2\frac{\lambda}{\rho}J_0(\frac{\rho}{\lambda})J_1(\frac{\rho}{\lambda})\sin^2(\frac{t}{\lambda})\right].$$

This gives $g(\lambda)$.

One-parameter family of solutions

Let $A_\lambda(t,\rho) = J_0\left(\frac{\rho}{\lambda}\right)\cos\left(\frac{t}{\lambda}\right)$ and $B_\lambda(t,\rho) = J_1\left(\frac{\rho}{\lambda}\right)\sin\left(\frac{t}{\lambda}\right)$. Then for $\lambda > 0$ the nonzero components of $T_{ab}(\lambda)$ are

$$
\begin{aligned}
T_{tt}(\lambda) &= T_{\rho\rho}(\lambda) = \frac{\alpha^2}{8\pi\lambda}(A_\lambda^2 + B_\lambda^2)\,, \\
T_{t\rho}(\lambda) &= T_{\rho t}(\lambda) = -\frac{\alpha^2}{4\pi\lambda}A_\lambda B_\lambda\,, \\
T_{\varphi\varphi}(\lambda) &= \frac{\alpha^2}{8\pi\lambda}\,\mathrm{e}^{-2\gamma_\lambda}\,\rho^2(A_\lambda^2 - B_\lambda^2)\,, \\
T_{zz}(\lambda) &= T_{\varphi\varphi}(\lambda)\,\rho^{-2}\,\mathrm{e}^{4\beta\sqrt{\lambda}F_\lambda}\,.
\end{aligned}
$$

Then

$$
\epsilon(\lambda) = \frac{1}{8\pi}\frac{\alpha^2}{\lambda}\,\mathrm{e}^{2(\beta\sqrt{\lambda}F_\lambda - \gamma_\lambda)}(A_\lambda^2 + B_\lambda^2)\,.
$$

Note the nontrivial behavior in the limit $\lambda \to 0$.

The background metric

For $\rho/\lambda \gg 1$:

$$J_n(\frac{\rho}{\lambda}) = \sqrt{\frac{2}{\pi}\frac{\lambda}{\rho}}\left[\cos(\rho/\lambda - \frac{\pi}{2}n - \frac{\pi}{4}) + O(\frac{\lambda}{\rho})\right] .$$

Using this we find the limit as $\lambda \to 0$:

$$\psi_\lambda \to 0\,, \quad \gamma_\lambda \to (\alpha^2 + \beta^2)\rho/\pi\,, \quad \phi_\lambda \to 0\,.$$

Thus

$$g^{(0)} = e^{2(\alpha^2+\beta^2)\rho/\pi}\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2\right) + \rho^2\mathrm{d}\varphi^2 + \mathrm{d}z^2\,.$$

The limiting functions do not satisfy one of the field equations, hence $g^{(0)}$ does not belong to the described class of solutions.

The background metric

The nonzero components of $G_{ab}(g^{(0)})$ are

$$G_{tt}(g^{(0)}) = G_{\rho\rho}(g^{(0)}) = \frac{\alpha^2 + \beta^2}{\pi\rho} \,.$$

In the weak limit the nonzero components of the scalar field energy-momentum tensor are

$$T_{tt}^{(0)} = T_{\rho\rho}^{(0)} = \frac{\alpha^2}{8\pi^2\rho} \,.$$

So, the effective energy-momentum tensor can be readily calculated:
$t_{ab}^{(0)} = \frac{1}{8\pi} G_{ab}(g^{(0)}) - T_{ab}^{(0)}$.

The μ tensor

$$
\begin{aligned}
&\mu_{tttttt} = \mu_{tt\rho\rho\rho\rho} = -\mu_{tttt\rho\rho} \\
&\quad = \mu_{\rho\rho tttt} = \mu_{\rho\rho\rho\rho\rho\rho} = -\mu_{\rho\rho tt\rho\rho} = \left[\frac{2}{\pi}\beta^2\rho^{-1} + \frac{1}{\pi^2}\left(\alpha^2+\beta^2\right)^2\right] e^{4(\alpha^2+\beta^2)\rho/\pi}\,, \\
&\mu_{tt\varphi\varphi\varphi\varphi} = \mu_{\rho\rho\varphi\varphi\varphi\varphi} = \frac{2}{\pi}\beta^2\rho^3\,, \\
&\mu_{ttzzzz} = \mu_{\rho\rho zzzz} = \frac{2}{\pi}\beta^2\rho^{-1}\,, \\
&\mu_{tt\rho\rho\varphi\varphi} = -\mu_{tttt\varphi\varphi} = \mu_{\rho\rho\rho\rho\varphi\varphi} = -\mu_{\rho\rho tt\varphi\varphi} = \frac{2}{\pi}\beta^2\rho\, e^{2(\alpha^2+\beta^2)\rho/\pi}\,, \\
&\mu_{tt\rho\rho zz} = -\mu_{ttttzz} = \mu_{\rho\rho\rho\rho zz} = -\mu_{\rho\rho ttzz} = -\frac{2}{\pi}\beta^2\rho^{-1} e^{2(\alpha^2+\beta^2)\rho/\pi}\,, \\
&\mu_{tt\varphi\varphi zz} = \mu_{\rho\rho\varphi\varphi zz} = -\frac{2}{\pi}\beta^2\rho\,.
\end{aligned}
$$

All other components follow from $\mu_{abcdef} = \mu_{(ab)(cd)(ef)} = \mu_{abefcd}$ or vanish.

The effective energy-momentum tensor

Nonzero components:

$$t_{tt}^{(0)} = t_{\rho\rho}^{(0)} = \frac{\beta^2}{8\pi^2\rho}\,.$$

Properties:

- $g(\lambda)$ satisfies all GW assumptions,
- $t^{(0)}$ satisfies all GW theorems (traceless, WEC),
- inhomogeneities of the scalar field do not contribute in the leading order to the backreaction effect (no dependence on α),
- for the chosen subclass of solutions $t^{(0)}$ is unique.