Axions in Gravity with Torsion

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Outline

Introduction: Why torsion?

Einstein (1915): General Relativity

- Torsion-free condition: $\mathcal{T}_{\mu}{}^{\lambda}{}_{\nu} = \Gamma_{\mu}{}^{\lambda}{}_{\nu} - \Gamma_{\nu}{}^{\lambda}{}_{\mu} \equiv 2\Gamma_{[\mu}{}^{\lambda}{}_{\nu]} = 0$
- *Local* Lorentz Symmetry and diffeomorphisms.
- One gravitation dynamical field: the metric.
- Second order equations of motion for the metric.

Cartan (1922): First order formalism

- The torsion-free condition is relaxed: $\mathcal{T}_{\mu}{}^{\lambda}{}_{\nu} \neq 0$.
- Equivalence with GR when torsion vanishes.
- Two gravitation dynamical fields: metric and torsion.
- First order equations of motion for the fields.

Introduction: Why torsion?

- Natural generalization of GR.
- It is consistent with the vacuum tests of GR.
- Torsion appears naturally in:
  - Gauge theories of gravitation,
  - String theory,
  - Supergravity, etc.
- The introduction of fermions is straightforward.
- Allows the solution of the strong CP problem.

Notation and conventions

- Vielbein: $g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu$ with $\eta_{ab} = \mathrm{diag}\,(-,+,\ldots,+)$.
- $V^a = e^a_\mu\, V^\mu$ and $V^\mu = E^\mu_a\, V^a$, where $e^a_\mu\, E^\mu_b = \delta^a_b$ and $e^a_\mu\, E^\nu_a = \delta^\nu_\mu$.
- Wedge Product (σ denotes the permutation of the indices)

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \sum_\sigma (-1)^{|\sigma|}\, dx^{\sigma(\mu_1)} \otimes \ldots \otimes dx^{\sigma(\mu_p)}, \tag{1}$$

- p-forms are defined ($\mathbf{e}^a \equiv e^a_\mu\, dx^\mu$)

$$\boldsymbol{\alpha} = \frac{1}{p!}\, \alpha_{\mu_1\ldots\mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \frac{1}{p!}\, \alpha_{a_1\ldots a_p}\, \mathbf{e}^{a_1} \wedge \ldots \wedge \mathbf{e}^{a_p}, \tag{2}$$

- D-dimensional Hodge dual

$$\star\left(\mathbf{e}^{a_1} \wedge \ldots \wedge \mathbf{e}^{a_p}\right) = \frac{1}{(D-p)!}\, \epsilon^{a_1\ldots a_p}{}_{a_{p+1}\ldots a_D}\, \mathbf{e}^{a_{p+1}} \wedge \ldots \wedge \mathbf{e}^{a_D}. \tag{3}$$

Notation and conventions

- Covariant derivative of a Lorentz tensor ($\mathbf{d} \equiv \mathbf{e}^a \partial_a$)

$$\boldsymbol{D}\boldsymbol{A}^a{}_b = \mathbf{d}\boldsymbol{A}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{A}^c{}_b - \boldsymbol{\omega}^c{}_b \wedge \boldsymbol{A}^a{}_c \,, \tag{4}$$

- Cartan structure equations

$$\mathbf{d}\mathbf{e}^a + \boldsymbol{\omega}^a{}_c \wedge \mathbf{e}^c = \boldsymbol{\mathcal{T}}^a = \frac{1}{2}\,\mathcal{T}_m{}^a{}_n\,\mathbf{e}^m \wedge \mathbf{e}^n, \tag{5}$$

$$\mathbf{d}\boldsymbol{\omega}^{ab} + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^{cb} = \boldsymbol{\mathcal{R}}^{ab} = \frac{1}{2}\,\mathcal{R}^{ab}{}_{mn}\,\mathbf{e}^m \wedge \mathbf{e}^n. \tag{6}$$

- Decomposition of the Lorentz connection $\boldsymbol{\omega}^{ab} = \mathring{\boldsymbol{\omega}}^{ab}(e) + \boldsymbol{\mathcal{K}}^{ab}$, where

$$\mathbf{d}\mathbf{e}^a + \mathring{\boldsymbol{\omega}}^a{}_b \wedge \mathbf{e}^b \equiv \mathring{\boldsymbol{D}}\mathbf{e}^a = 0 \quad \text{and} \quad \boldsymbol{\mathcal{T}}^a = \boldsymbol{\mathcal{K}}^a{}_b \wedge \mathbf{e}^b\,. \tag{7}$$

- Bianchi identities

$$\boldsymbol{D}\boldsymbol{\mathcal{T}}^a = \boldsymbol{\mathcal{R}}^a{}_b \wedge \mathbf{e}^b \qquad \text{and} \qquad \boldsymbol{D}\boldsymbol{\mathcal{R}}^{ab} = 0 \tag{8}$$

Actions in physics

- Scalar field

$$S_\phi = -\frac{1}{2}\int \mathbf{d}\phi \wedge \star\mathbf{d}\phi = -\frac{1}{2}\int d^D x\, e\, \partial_m \phi\, \partial^m \phi$$

- Dirac field

$$S_\psi = -\frac{1}{2}\int \left(\bar{\psi}\boldsymbol{\gamma} \wedge \star\boldsymbol{D}\psi + \text{h.c.}\right) = -\frac{1}{2}\int d^D x\, e\, E_a^\mu \left(\bar{\psi}\gamma^a\, D_\mu \psi + \text{h.c.}\right)$$

- Yang-Mills

$$S_{\text{YM}} = -\int \text{Tr}\left[\boldsymbol{F} \wedge \star\boldsymbol{F}\right] = -\frac{1}{4}\int d^D x\, e\, F^A_{\mu\nu}\, F^{A\,\mu\nu}$$

- Einstein-Cartan

$$S_{\text{gr}} = \frac{1}{2\kappa^2}\int \boldsymbol{\mathcal{R}}_{ab} \wedge \star\left(\mathbf{e}^a \wedge \mathbf{e}^b\right) = \frac{1}{2\kappa^2}\int d^D x\, e\, \mathcal{R}$$

Einstein-Cartan Theory

- Metricity condition, i.e. $\nabla_\rho g_{\mu\nu} = 0$.
- The connection $\Gamma_\mu{}^\lambda{}_\nu = \left\{{}_\mu{}^\lambda{}_\nu\right\} + \mathcal{K}_\mu{}^\lambda{}_\nu$ includes an additional piece, called contorsion, which encodes the torsional information through

$$\mathcal{K}_\mu{}^\lambda{}_\nu = \frac{1}{2}\left(\mathcal{T}_\mu{}^\lambda{}_\nu - \mathcal{T}_{\nu\mu}{}^\lambda + \mathcal{T}^\lambda{}_{\nu\mu}\right). \tag{9}$$

- Metric and contorsion are independent fields.
- Einstein-Cartan action

$$S_{\text{gr}} = \frac{1}{2\kappa^2}\int \boldsymbol{\mathcal{R}}_{ab} \wedge \star\left(\mathbf{e}^a \wedge \mathbf{e}^b\right) = \frac{1}{2\kappa^2}\int d^4x\, e\, \mathcal{R}\,, \tag{10}$$

where we have defined $e = \det e^a_\mu$ and $\mathcal{R} = E^a_\mu\, E^b_\nu\, \mathcal{R}^{ab}{}_{\mu\nu}$ with

$$\mathcal{R}^{ab}{}_{\mu\nu} = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu|}{}^a{}_c\,\omega_{|\nu]}{}^{cb}\,, \tag{11}$$

and $\omega_\mu{}^a{}_b = e^a_\lambda\left(\partial_\mu E^\lambda_b + \Gamma_\mu{}^\lambda{}_\nu\, E^\nu_b\right)$.

Einstein-Cartan Theory in vacuum

- The D-dimensional Einstein-Cartan theory in vacuum is described by the action

$$S = \frac{1}{2\kappa_*^2}\int \boldsymbol{\mathcal{R}}_{ab}\wedge\star\left(\mathbf{e}^a\wedge\mathbf{e}^b\right)\ . \tag{12}$$

The equations of motion within Cartan's formalism

$$\delta\mathbf{e}^a \quad : \quad \mathcal{R}_{ab} - \frac{1}{2}\eta_{ab}\,\mathcal{R} = 0\,, \tag{13}$$

$$\delta\boldsymbol{\omega}^{ab} \quad : \quad \mathcal{T}_a{}^p{}_b - 2\mathcal{T}_{[a}\delta^p_{b]} = 0\,. \tag{14}$$

The solution to the algebraic Eq. (14) is $\mathcal{T}_{abc} = 0$. Thus, $\boldsymbol{\omega}^{ab} = \mathring{\boldsymbol{\omega}}^{ab}(e)$ and the Eq. (13) reduces to the usual Einstein's equations in vacuum.

Einstein-Cartan Theory coupled with fermions

- D-dimensional action[1]

$$S = \frac{1}{2\kappa_*^2}\int \boldsymbol{\mathcal{R}}_{ab}\wedge\star\left(\mathbf{e}^a\wedge\mathbf{e}^b\right) + \frac{1}{2}\int\left(\bar{\psi}\boldsymbol{\gamma}\wedge\star\boldsymbol{D}\psi - \boldsymbol{D}\bar{\psi}\wedge\star\boldsymbol{\gamma}\psi\right) \tag{15}$$

where $\kappa_*^2 \sim \frac{1}{M_*^{2+n}}$, $\boldsymbol{\gamma} = \gamma_a\mathbf{e}^a$, $\bar{\psi} = -\imath\psi^\dagger\gamma^0$ and the covariant derivative[2]

$$\boldsymbol{D}\psi = \mathbf{d}\psi + \frac{1}{4}\boldsymbol{\omega}^{ab}\gamma_{ab}\psi. \tag{16}$$

- Decomposing the Lorentz connection $\boldsymbol{\omega}^{ab} = \mathring{\boldsymbol{\omega}}^{ab}(e) + \boldsymbol{\mathcal{K}}^{ab}$, we obtain an equivalent action (up-to-a boundary term)

$$S = \frac{1}{2\kappa_*^2}\int \mathring{\boldsymbol{\mathcal{R}}}_{ab}\wedge\star\left(\mathbf{e}^a\wedge\mathbf{e}^b\right) - \frac{1}{2}\int\left(\bar{\psi}\boldsymbol{\gamma}\wedge\star\mathring{\boldsymbol{D}}\psi - \mathring{\boldsymbol{D}}\bar{\psi}\wedge\star\boldsymbol{\gamma}\psi\right)$$
$$+ \frac{1}{2\kappa_*^2}\int \boldsymbol{\mathcal{K}}_{am}\wedge\boldsymbol{\mathcal{K}}^m{}_b\wedge\star\left(\mathbf{e}^a\wedge\mathbf{e}^b\right) - \frac{1}{8}\int\boldsymbol{\mathcal{K}}^{ab}\wedge\star\bar{\psi}\left\{\boldsymbol{\gamma},\gamma_{ab}\right\}\psi \tag{17}$$

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[1] $D = 4 + n$ and the sum over the fermionic flavour is assumed.

[2] $\gamma_{a_1\ldots a_n} \equiv \gamma_{[a_1}\cdots\gamma_{a_n]}$.

Einstein-Cartan Theory coupled with fermions

Equations of motion within the Cartan's formalism

$$\delta \mathbf{e}^a \quad : \quad \mathcal{R}_{ab} - \frac{1}{2}\eta_{ab}\mathcal{R} = \kappa_*^2\, \tau_{ab} \tag{18}$$

$$\delta \boldsymbol{\omega}^{ab} \quad : \quad \mathcal{T}_a{}^b{}_c - 2\mathcal{T}_{[a}\delta^b_{c]} = -\frac{\kappa_*^2}{2}\,\bar{\psi}\gamma_a{}^b{}_c\psi \tag{19}$$

$$\delta \bar{\psi} \quad : \quad \gamma^a \mathring{D}_a\psi + \frac{1}{4}\mathcal{K}_{abc}\gamma^{abc}\psi = 0 \tag{20}$$

Solving the algebraic Eq. (19) we obtain

$$\mathcal{K}_{abc} = -\frac{\kappa_*^2}{4}\,\bar{\psi}\gamma_{abc}\psi, \tag{21}$$

for the contorsion tensor. Replacing it back into the initial action, leads to the D-dimensional effective theory

$$S = \mathring{S}_{\text{gr}} + \mathring{S}_{\psi} + \frac{\kappa_*^2}{32}\int d^Dx\, e\,\bar{\psi}\gamma_{abc}\psi\,\bar{\psi}\gamma^{abc}\psi. \tag{22}$$

Einstein-Cartan Theory coupled with fermions

- In the $D = 4$ case, we have the identity

$$\bar{\psi}\gamma_{abc}\psi = \imath\,\epsilon_{abcd}\,\bar{\psi}\gamma^{d}\gamma_5\psi\,, \tag{23}$$

where $\gamma_5 = \imath\,\gamma_0\gamma_1\gamma_2\gamma_3$. Defining $J_5^a = \imath\,\bar{\psi}\gamma^a\gamma_5\psi$ and $\boldsymbol{J}_5 = J_5^a\,\boldsymbol{e}_a$, the torsion-induced four-fermion interaction in 4-dimensions is

$$S = \mathring{S}_{\text{gr}} + \mathring{S}_{\psi} - \frac{3\kappa^2}{16}\int \boldsymbol{J}_5 \wedge \star\boldsymbol{J}_5\,. \tag{24}$$

Since $\kappa^2 \sim 10^{-36}\,\text{GeV}^{-2}$ in 4-dimensions, the torsion-induced four-fermion interaction is highly suppressed in such scenario.

Gauge invariance

- An invariant $SU(N)$ gauge theory is obtained from the transformations

$$\psi \to e^{\imath\, \theta^A(x)\, T_A}\, \psi \quad \text{and} \quad \boldsymbol{\mathcal{D}}\psi \to e^{\imath\, \theta^A(x)\, T_A}\, \boldsymbol{\mathcal{D}}\psi\,, \tag{25}$$

where T_A are the generators of the $SU(N)$ Lie group.[3] The fermionic covariant derivative includes gauge fields $\boldsymbol{\mathcal{B}} = T_A \mathcal{B}^A_\mu\, dx^\mu$ as

$$\boldsymbol{\mathcal{D}}\psi = \mathbf{d}\psi + \frac{1}{4}\boldsymbol{\omega}^{ab}\gamma_{ab}\psi + \imath\, g\, \boldsymbol{\mathcal{B}}\psi = \boldsymbol{D}\psi + \imath\, g\, \boldsymbol{\mathcal{B}}\psi\,. \tag{26}$$

The transformation of the covariant derivative in Eq. (25) is guaranteed if the gauge field transforms as an $SU(N)$ connection

$$\mathcal{B}^A_\mu \to \mathcal{B}^A_\mu - f_{MN}{}^{A}\,\theta^M\,\mathcal{B}^N_\mu - \frac{1}{g}\,\partial_\mu\theta^A\,. \tag{27}$$

[3] They satisfy the Lie algebra $[T_A, T_B] = \imath\, f_{AB}{}^{C}\, T_C$.

Strong CP problem

- Addition of the θ-term to the QCD Lagrangian

$$\mathscr{L}_{\text{QCD}} \supset -\theta \frac{\alpha_s}{2\pi} \operatorname{Tr}(\boldsymbol{G} \wedge \boldsymbol{G}) \tag{28}$$

Strong CP problem

Limits on the neutron's electric dipole moment $\rightarrow \theta \leq 10^{-10}$

Peccei and Quinn solution (1977)

- Extra $U(1)_A$ symmetry, spontaneously broken at $\sim \Lambda_{EW}$.
- Axion coupled to Pontryagin density, i.e. $\sim \phi(x) \operatorname{Tr}[\boldsymbol{G} \wedge \boldsymbol{G}]$.
- Promote $\theta \rightarrow \theta(x) \sim \theta + \phi(x)/f_\phi$ with $\langle\phi\rangle = -f_\phi \theta$.
- Perturbations around $\langle\phi\rangle$ gives a CP-even $a(x) \operatorname{Tr}[\boldsymbol{G} \wedge \boldsymbol{G}]$.

Axions in gravity with torsion

- Motivation: QFT in background geometry
- $SU(N) \times U(1)$ gauge invariant action coupled with fermions

$$S = \frac{1}{2\kappa_*^2} \int \boldsymbol{\mathcal{R}}_{ab} \wedge \star \left(\mathbf{e}^a \wedge \mathbf{e}^b\right) + \frac{1}{2} \int \left(\bar{\psi} \boldsymbol{\gamma} \wedge \star \boldsymbol{\mathcal{D}} \psi - \boldsymbol{\mathcal{D}} \bar{\psi} \wedge \star \boldsymbol{\gamma} \psi\right)$$
$$- \frac{1}{2} \int \boldsymbol{F} \wedge \star \boldsymbol{F} - \int \mathrm{Tr}\left[\boldsymbol{G} \wedge \star \boldsymbol{G}\right] - \theta \frac{\alpha_s}{2\pi} \int \mathrm{Tr}\left[\boldsymbol{G} \wedge \boldsymbol{G}\right] \tag{29}$$

- **Duncan et.al:** *Nucl.Phys.B387,215 (1992)*
  - Impose the classical conservation $\mathbf{d} \star \boldsymbol{\mathcal{S}} = 0$, where $\star \boldsymbol{\mathcal{S}} = \mathbf{e}^a \wedge \boldsymbol{\mathcal{T}}_a$, at quantum level through

$$\mathcal{Z} = \int \prod_{\varphi} \mathcal{D}\varphi \, \mathcal{D}\boldsymbol{\mathcal{S}} \, e^{\imath S[\varphi, \boldsymbol{\mathcal{S}}]} \int \mathcal{D}\phi \, e^{\imath \int \phi \, \mathbf{d} \star \boldsymbol{\mathcal{S}}}. \tag{30}$$

Axions in gravity with torsion

- **Mielke and Sánchez:** *Phys.Rev.D73,043521 (2006)*
  - Argue the appearance of $\mathbf{d}\boldsymbol{\mathcal{S}} \wedge \mathbf{d}\boldsymbol{\mathcal{S}}$ in the axial anomaly.
  - Modified axial-current by the addition of Chern-Simons-type terms

$$\star\hat{\boldsymbol{J}}_5 = \star\boldsymbol{J}_5 + \frac{\alpha_{\mathrm{em}}\bar{Q}^2}{\pi}\,\boldsymbol{C}_{FF} + \frac{\alpha_s N_q}{2\pi}\,\boldsymbol{C}_{GG} + \frac{N_f}{8\pi^2}\left(\boldsymbol{C}_{RR} + \boldsymbol{\mathcal{S}} \wedge \mathbf{d}\boldsymbol{\mathcal{S}}\right) .$$

  - The conservation of the modified axial-current occurs when $\boldsymbol{\mathcal{S}} \sim \mathbf{d}\phi$, where ϕ is a pseudo-scalar potential.

- **Mercuri:** *Phys.Rev.Lett.103,081302 (2009)*
  - Divergent Nieh-Yan term in the $U(1)_A$ rotated fermionic measure.[4]
  - Add to the action (29) the topological Nieh-Yan density, i.e.

$$S \to S + \beta \int \left(\boldsymbol{\mathcal{T}}^a \wedge \boldsymbol{\mathcal{T}}_a - \boldsymbol{\mathcal{R}}_{ab} \wedge \mathbf{e}^a \wedge \mathbf{e}^b\right) = S + \beta \int \mathbf{d}\left(\mathbf{e}^a \wedge \boldsymbol{\mathcal{T}}_a\right).$$

  - Promote the BI parameter to be a field, i.e. $\beta \to \beta(x)$ and absorb the divergence by means of renormalized $\beta(x)$.

[4] Chandía & Zanelli *Phys.Rev.D55,7580 (1997).*

Axions in gravity with torsion

- Integrating out the torsion in either of these approaches leads to[5]

$$S_{\text{eff}} = S_0 + S_\theta - \frac{1}{2f_\Phi^2}\int \boldsymbol{J}_5 \wedge \star \boldsymbol{J}_5 - \frac{1}{2}\int \mathbf{d}\Phi \wedge \star \mathbf{d}\Phi + \frac{1}{f_\Phi}\int \Phi\, \mathbf{d} \star \boldsymbol{J}_5 \,.$$

- Either approach gives a vanishing Nieh-Yan density.
- Replacing the axial anomaly

$$\mathbf{d} \star \boldsymbol{J}_5 = -\frac{\alpha_{\text{em}}\,\bar{Q}^2}{\pi}\boldsymbol{F}\wedge\boldsymbol{F} - \frac{\alpha_s N_q}{2\pi}\,\text{Tr}\left[\boldsymbol{G}\wedge\boldsymbol{G}\right] - \frac{N_f}{8\pi^2}\mathring{\boldsymbol{\mathcal{R}}}^{ab}\wedge\mathring{\boldsymbol{\mathcal{R}}}_{ab}\,, \qquad (31)$$

gives the explicit form of the effective theory[6]

$$S_{\text{eff}} = S_0 - \frac{1}{2f_\Phi^2}\int \boldsymbol{J}_5 \wedge \star \boldsymbol{J}_5 - \frac{\alpha_{\text{em}}\,\bar{Q}^2}{\pi f_\Phi}\int \Phi\,\boldsymbol{F}\wedge\boldsymbol{F} - \frac{1}{2}\int \mathbf{d}\Phi \wedge \star \mathbf{d}\Phi$$
$$- \frac{1}{8\pi^2}\int \left(\Theta + \frac{N_f}{f_\Phi}\,\Phi\right)\mathring{\boldsymbol{\mathcal{R}}}^{ab}\wedge\mathring{\boldsymbol{\mathcal{R}}}_{ab} - \frac{\alpha_s}{2\pi}\int \left(\theta + \frac{N_q}{f_\Phi}\,\Phi\right)\text{Tr}\left[\boldsymbol{G}\wedge\boldsymbol{G}\right].$$

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[5] We have defined $S_0 = \mathring{S}_{\text{gr}} + \mathring{S}_\psi + S_{\text{gk}}$, $f_\Phi = \kappa^{-1}\sqrt{8/3}$ and $\Phi = 4/3 f_\Phi^{-1}\vartheta$, where $\vartheta = \beta, \phi$.

[6] N_f: number of fermionic flavors, N_q: number of quarks and $\bar{Q}^2 = \sum_f Q_f^2$.

Axions in gravity with torsion

Parameters

	$M_{Pl} \sim 10^{18}$ [GeV]	$M_* \sim 10^4$ [GeV]	$M_* \sim 10^2$ [GeV]
f_Φ [GeV]	10^{18}	10^4	10^2
m_a [keV]	10^{-15}	10^{-1}	10
$\Gamma_{a\to\gamma\gamma}$ [keV]	10^{-101}	10^{-32}	10^{-19}

- Torsion-descended axions as the dominant dark matter content[7]

$$r \leq 1.6 \times 10^{-9}. \tag{32}$$

[7] M. Lattanzi & S. Mercuri *Phys.Rev.D81,125015.*

Conclusions and outlook

- Einstein-Cartan + fermions $\rightarrow$ four-fermion interaction.
- Suitable modifications to such a theory solves the strong CP problem.
- Rather different motivations for the torsion-descended axions leads to the same effective theory.
- The axionic phenomenology is characterized only by the gravitational scale.
- Torsion-descended axions might be dark matter candidates.

Outlook

- Study their role in extra-dimensional scenarios.
- Include them in more general theories of gravitation.

Thank you!