Internal clock formulation of quantum mechanics

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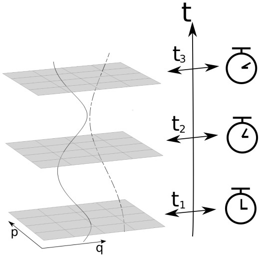
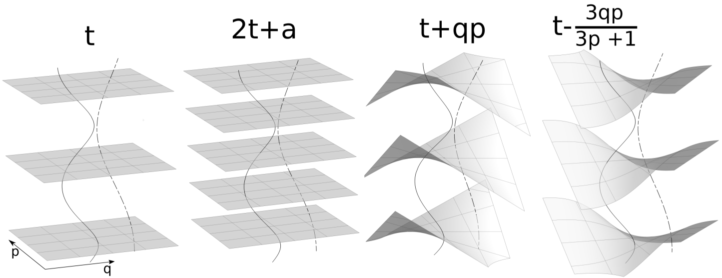
Outline

1. Definition of internal clock
2. Internal clock in quantisation
3. Properties of new quantum mechanics
4. Limit of ordinary quantum mechanics
5. Properties of semiclassical dynamics

Internal clock



contact manifold $M_C = \textit{phase space} \times \mathbb{R}$



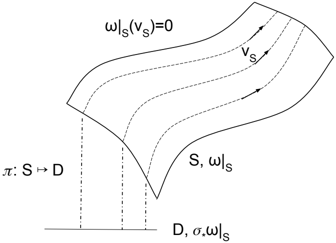
Internal clock

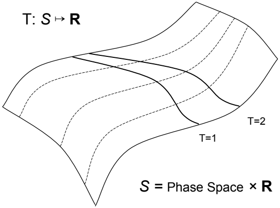
$$N \cdot H(q^i, p_j) = 0 \tag{1}$$

$$N \cdot H(q^i, p_j) = p^1 + h(q^1, q^2, p_2, q^3, p_3, \dots) \tag{2}$$

$$(q^k, p_k), \quad k = 2, 3, \dots, \quad t \equiv q^1, \quad h(t, q^k, p^k) \tag{3}$$

HUGE ambiguity!





Canonical formalism

Canonical transformations $(q^I, p_I, t) \mapsto (\bar{q}^I, \bar{p}_I)$:

$$\omega_{\mathcal{C}} = \mathrm{d}q^I \mathrm{d}p_I - \mathrm{d}t \mathrm{d}h = \mathrm{d}\bar{q}^I \mathrm{d}\bar{p}_I - \mathrm{d}t \mathrm{d}\bar{h} \tag{4}$$

Pseudocanonical transformations $(q^I, p_I, t) \mapsto (\bar{q}^I, \bar{p}_I, \bar{t})$:

$$\omega_{\mathcal{C}} = \mathrm{d}q^I \mathrm{d}p_I - \mathrm{d}t \mathrm{d}h = \mathrm{d}\bar{q}^I \mathrm{d}\bar{p}_I - \mathrm{d}\bar{t} \mathrm{d}\bar{h} \tag{5}$$

Note the definition of the symplectic form as $\omega_{\mathcal{C}}|_t$.

Clock transformations form a group $\mathcal{G}_{clock}$ with canonical transformations $\mathcal{G}_{can}$ as its normal subgroup $\Rightarrow$ fibre bundle $\pi: \mathcal{G}_{clock} \to \mathcal{T}$ over the space of internal clocks $\mathcal{T}$ with canonical transformations $\mathcal{G}_{can}$ as a fibre.

Canonical formalism

Let us consider a section:

$$\sigma : \mathcal{T} \ni t \mapsto (q, p, t) \in \mathcal{G}_{clock} \tag{6}$$

such that

$C_I(t, q, p)$ is a Dirac observable $\Leftrightarrow$ $C_I(\bar{t}, \bar{q}, \bar{p})$ is a Dirac observable (i.e. a conserved quantity)

Specify the section σ by means of $2n + 1$ algebraic equations:

$$\bar{t} = \bar{t}(t, q, p),\ \ C_I(t, q, p) = C_I(\bar{t}, \bar{q}, \bar{p}),\ \ I = 1, \ldots, 2n$$

Example

Consider the contact form:

$$\omega_C = \mathrm{d}q\mathrm{d}p - \mathrm{d}t\mathrm{d}\mathbf{H}, \quad \mathbf{H} = \frac{p^2}{2}, \quad (q,p) \in \mathbb{R}^2, \quad t \in \mathbb{R} \tag{7}$$

Dirac observables are

$$C_1(q,p,t) = p, \quad C_2(q,p,t) = q - pt \tag{8}$$

and the special pseudocanonical transformation is given by

$$\bar{t} = t + D(q,p), \;\; \bar{p} = p, \;\; \bar{q} = q - pD(q,p), \tag{9}$$

The contact form reads now:

$$\omega_C = \mathrm{d}\bar{q}\mathrm{d}\bar{p} - \mathrm{d}\bar{t}\mathrm{d}\mathbf{H}, \quad \mathbf{H} = \frac{\bar{p}^2}{2}, \quad (\bar{q},\bar{p}) \in \mathbb{R}^2, \quad \bar{t} \in \mathbb{R} \tag{10}$$

Quantisation of all clock-frames

Quantisation is assumed to be a linear map of the form

$$f(q,p,t) \mapsto \hat{A}_f := \int_{t=const} \mathrm{d}q\mathrm{d}p\ f(q,p,t) M(q,p), \qquad (11)$$

where $M(q,p)$ is a family of bounded operators on $\mathcal{H}$ such that $\int \mathrm{d}q\mathrm{d}p\ M(q,p) = \mathbb{I}_{\mathcal{H}}$. E.g. for the "canonical prescription",

$$M(q,p) = \mathbf{D}(q,p) 2\mathcal{P} \mathbf{D}^{\dagger}(q,p), \quad \mathbf{D}(q,p) = e^{i(p\hat{Q} - q\hat{P})} \qquad (12)$$

For all choices of internal clock assign to Dirac observables the same quantum representation on a fixed $\mathcal{H}$.

Quantisation of all observables in all internal clocks is completely fixed by the Dirac observables' representation.

Properties of quantised clock-frames

0) Any physical state is represented by a unique vector

$$|\Psi\rangle \in \mathcal{H}$$

1) Any Dirac observable, $C(q, p, t) = C(\bar{q}, \bar{p}, \bar{t})$, is promoted to a unique operator

$$C \mapsto \hat{C}, \quad \Psi(c) := \langle \phi_c | \Psi \rangle \in L^2(sp(\hat{C}), \mathrm{d}c)$$

2) For any dynamical observable, $D(q, p, t) = \bar{D}(\bar{q}, \bar{p}, \bar{t})$, the respective operator depend son the choice of internal clock

$$D \mapsto \hat{D} \quad \text{and} \quad \bar{D} \mapsto \hat{\bar{D}} \neq \hat{D}$$

$$\Psi(d) := \langle \phi_d | \Psi \rangle \in L^2(sp(\hat{D}), \mathrm{d}d)$$

3) There is a unique Schrödinger equation

$$i\partial_\tau |\Psi\rangle = \hat{C}|\Psi\rangle, \quad \{\tau\} \in \mathcal{T},$$

and thus, the evolution is independent of the choice of clock.

Example

$$\omega_C = \mathrm{d}q\mathrm{d}p - \mathrm{d}t\mathrm{d}\mathbf{H}, \quad \mathbf{H} = \frac{p^2}{2}, \quad (q,p) \in \mathbb{R}^2, \quad t \in \mathbb{R} \tag{13}$$

$$t = \bar{t} + D(\bar{q}, \bar{p}), \quad p = \bar{p}, \; q = \bar{q} - \bar{p}D(\bar{q}, \bar{p}). \tag{14}$$

Quantisation of $p \mapsto \hat{P}$ is unique and of q is ambiguous,

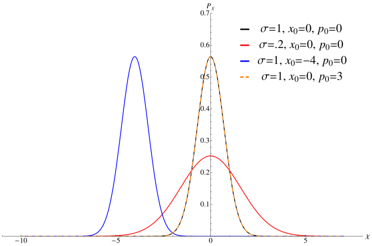
$$q \mapsto Sym[\hat{Q} - \hat{P}D(\hat{Q}, \hat{P})]. \tag{15}$$

Example

Set $D(\bar{q}, \bar{p}) = \bar{q}\bar{p}$. Then in momentum repr.:

$$q \mapsto i(1+p^2)\frac{\partial}{\partial p} + ip, \quad |q\rangle = \frac{e^{-iq\arctan(p)}}{\sqrt{\pi}\sqrt{p^2+1}}, \quad q = 2n + \mu$$

Fix $\Psi(p) = \langle p|\Psi\rangle = (\pi\sigma)^{-1/4} e^{-\frac{1}{2\sigma}(p-p_0)^2} e^{-ix_0 p}$. What is $|\langle q|\Psi\rangle|^2$?



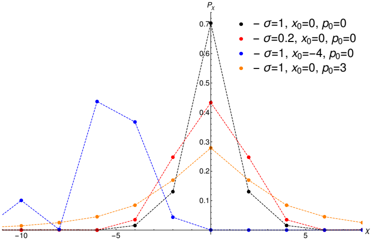


Figure: Probability distribution $P_q = |\langle q|\Psi\rangle|^2$ of position eigenvalues for the state $|\Psi\rangle$ in the clock t (on the left) and in the clock $\bar{t} = t - \bar{q}\bar{p}$ (on the right). The clock transformation turns the real spectrum into a discrete one.

Limit of ordinary QM: internal observer

Let the entire system be the product of **system** and **observer**:

$$(q_s, p_s, q_o, p_o) \in \mathbb{R}^4, \quad t \in \mathbb{R}, \tag{16}$$

$$\omega = \omega_s + \omega_o, \quad \omega_i = \mathrm{d}q_i \mathrm{d}p_i - \mathrm{d}t \mathrm{d}H_i, \quad H_i = \frac{p_i^2}{2}, \quad i = s, o$$

Let the clock transformation involve **observer** only:

$$t \mapsto \bar{t} = t + D(q_o, p_o). \tag{17}$$

observer: $\omega_o|_{\bar{t}} \neq \omega_o|_t$,

system: $\omega_s|_{\bar{t}} = \omega_s|_{t+\Delta(t)}, \quad \Delta(t) = D(q_s(t), p_s(t))$

t- and $\bar{t}$-frames of quantum **system** are related by $U = e^{-\frac{i\Delta}{2}P^2}$:

Clock t	Clock $\bar{t} = t + \Delta(t)$
$p_s \mapsto \hat{P}$ $q_s \mapsto \hat{Q}$ $\lvert\Psi\rangle \mapsto \Psi(q) = \langle q\vert\Psi\rangle$ $i\partial_t \psi(q) = \hat{H}\psi(q)$	$p_s \mapsto \hat{P}$ $q_s \mapsto \hat{Q} - \Delta(t)\hat{P}$ $\lvert\Psi\rangle \mapsto \varphi(q) = \langle q\vert U^\dagger\vert\Psi\rangle$ $i\partial_{\bar{t}} \varphi(q) = \hat{H}\varphi(q)$

Semiclassical dynamics

Spacetime $\mathcal{M} = \mathbb{T}^3 \times \mathbb{R}$:

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + q^2[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2]. \tag{18}$$

with a perfect fluid, $\mathrm{p} = w\rho, \;\; w < 1$.
The Hamiltonian constraint:

$$N \cdot C^0 = p_T + \frac{c_w^2}{24} p^2, \quad q > 0, \tag{19}$$

$$\omega = \mathrm{d}q \mathrm{d}p + \mathrm{d}T \mathrm{d}p_T, \tag{20}$$

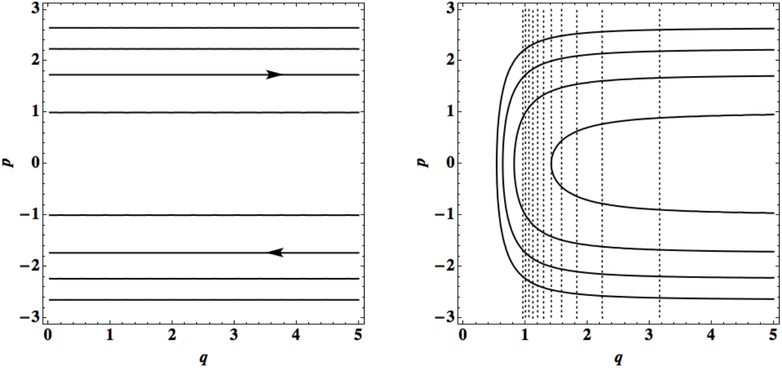
where (q, p) - isotropic geometry, (T, p_T) - perfect fluid.

C^0 **is solved by removing** p_T. The contact reads:

$$\omega|_{C^0=0} = \mathrm{d}q \mathrm{d}p - \mathrm{d}T \mathrm{d}\mathrm{H}, \quad \mathrm{H} = \frac{c_w^2}{24} p^2.$$

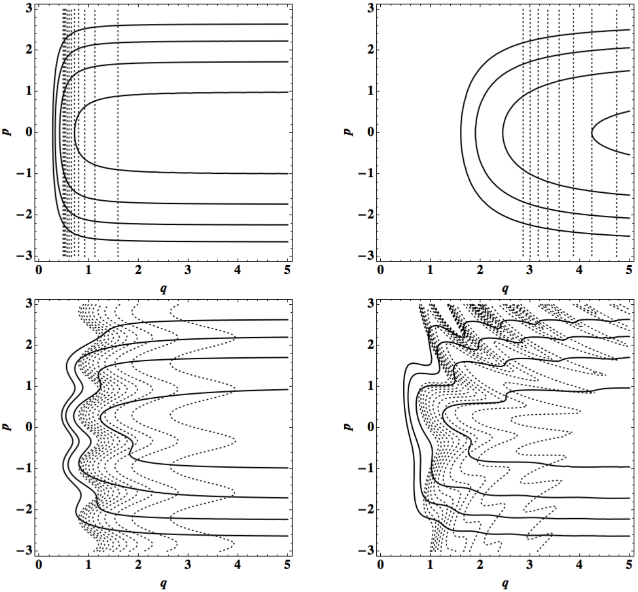
The canonical formalism of a free particle on the half-line.

Semiclassical dynamics



Left: classical trajectories, $H = p^2$.
Right: semiclassical trajectories, $H_{sem} = p^2 + \hbar^2 \frac{K}{q^2}$.

Semiclassical dynamics



Conclusions

- We choose an internal clock and make transformations thereof the SYMMETRY of the canonical formalism
- In internal clock formulation quantum states admit an unambiguous non-dynamical interpretation and many dynamical interpretations, but…
- …the ordinary formulation is regained as a special case with an internal observer
- …unambiguous predictions for semiclassical dynamics are possible.

Works

1. P. Małkiewicz, A. Miroszewski, Internal clock formulation of quantum mechanics, Phys. Rev. D 96 (2017) 046003
2. P. Małkiewicz, What is dynamics in quantum gravity?, Class. Quantum Grav. 34 (2017) 205001
3. P. Małkiewicz, Clocks and dynamics in quantum models of gravity, Class. Quantum Grav. 34 (2017) 145012
4. P. Małkiewicz, Multiple choices of time in quantum cosmology, Class. Quantum Grav. 32 (2015) 135004