Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 994 (2023) 116326

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

High Energy Physics – Theory

How Clifford algebra helps understand second quantized quarks and leptons and corresponding vector and scalar boson fields, *opening a new step beyond the standard model*

N.S. Mankoč Borštnik

Department of Physics, University of Ljubljana, SI-1000 Ljubljana, Slovenia

Received 30 April 2023; received in revised form 2 August 2023; accepted 5 August 2023
Available online 10 August 2023
Editor: Stephan Stieberger

Abstract

This article presents the description of the internal spaces of fermion and boson fields in d-dimensional spaces, with the odd and even "basis vectors" which are the superposition of odd and even products of operators γ^a. While the Clifford odd "basis vectors" manifest properties of fermion fields, appearing in families, the Clifford even "basis vectors" demonstrate properties of the corresponding gauge fields. In $d \geq (13+1)$ the corresponding creation operators manifest in $d = (3+1)$ the properties of all the observed quarks and leptons, with the families included, and of their gauge boson fields, with the scalar fields included, making several predictions. The properties of the creation and annihilation operators for fermion and boson fields are illustrated on the case $d = (5+1)$, when $SO(5,1)$ demonstrates the symmetry of $SU(3) \times U(1)$.

1. Introduction

The *standard model* (corrected with the right-handed neutrinos) has been experimentally confirmed without raising any severe doubts so far on its assumptions, which, however, remain unexplained.

E-mail address: norma.mankoc@fmf.uni-lj.si.

https://doi.org/10.1016/j.nuclphysb.2023.116326
0550-3213/

The *standard model* assumptions have several explanations in the literature, mostly with several new, not explained assumptions. The most popular are the grand unifying theories ([1–5] and many others).

In a long series of works ([6–9], and the references therein) the author has found, together with the collaborators ([10–14,23] and the references therein), the phenomenological success with the model named the *spin-charge-family* theory with the properties:

a. The internal space of fermions are described by the "basis vectors" which are superposition of odd products of anti-commuting objects (operators)[1] γ^a (in the sense $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$), Sect. 2.1, in $d = (13+1)$-dimensional space [13,14]. Correspondingly the "basis vectors" of one Lorentz irreducible representation in internal space of fermions, together with their Hermitian conjugated partners, anti-commute, fulfilling (on the vacuum state) all the requirements for the second quantized fermion fields ([10,14] and references therein).

a.i. The second kind of anti-commuting objects, $\tilde{\gamma}^a$, Sect. 2.1, equip each irreducible representation of odd "basis vectors" with the family quantum number [13,10].

a.ii. Creation operators for single fermion states — which are tensor products, $*_T$, of a finite number of odd "basis vectors" appearing in $2^{\frac{d}{2}-1}$ families, each family with $2^{\frac{d}{2}-1}$ members, and the (continuously) infinite momentum/coordinate basis applying on the vacuum state [13,14] — inherit anti-commutativity of "basis vectors". Creation operators and their Hermitian conjugated partners correspondingly anti-commute.

a.iii. The Hilbert space of second quantized fermion field is represented by the tensor products, $*_{T_H}$, of all possible numbers of creation operators, from zero to infinity [14], applying on a vacuum state.

a.iv. Spins from higher dimensions, $d > (3+1)$, described by the eigenvectors of the superposition of the Cartan subalgebra S^{ab}, Table 4, manifest in $d = (3+1)$ all the charges of the *standard model* quarks and leptons and antiquarks and antileptons.

b. In a simple starting action, Eq. (1), massless fermions carry only spins and interact with only gravity — with the vielbeins and the two kinds of spin connection fields (the gauge fields of momenta, of $S^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$ and of $\tilde{S}^{ab} = \frac{1}{4}(\tilde{\gamma}^a\tilde{\gamma}^b - \tilde{\gamma}^b\tilde{\gamma}^a)$, respectively[2]). The starting action includes only even products of γ^a's and $\tilde{\gamma}^a$'s ([14] and references therein).

b.i. Gravity — the gauge fields of S^{ab}, $((a, b) = (5, 6, ..., d))$, with the space index $m = (0, 1, 2, 3)$ — manifest as the *standard model* vector gauge fields [11], with the ordinary gravity included $((a, b) = (0, 1, 2, 3))$.

b.ii. The scalar gauge fields of $\tilde{S}^{ab}$, and of some of the superposition of S^{ab}, with the space index $s = (7, 8)$ manifest as the scalar Higgs and Yukawa couplings [9,14,23], determining mass matrices (of particular symmetry) and correspondingly the masses of quarks and leptons and of the weak boson fields after (some of) the scalar fields with the space index $(7, 8)$ gain constant values.

b.iii. The scalar gauge fields of $\tilde{S}^{ab}$ and of S^{ab} with the space index $s = (9, 10, ..., 14)$ and $(a, b) = (5, 6, ..., d)$ offer the explanation for the observed matter/antimatter asymmetry [8,9,12,14] in the universe.

c. The theory predicts at low energy two groups with four families. To the lower group of four families the so far observed three belong [32–38], and the stable of the upper four families, the

[1] According to Eq. (6) $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$ are anticommuting unless $a = b$.

[2] If no fermions are present, the two kinds of spin connection fields are uniquely expressible by the vielbeins.

fifth family of (heavy) quarks and leptons, offers the explanation for the appearance of dark matter. Due to the heavy masses of the fifth family quarks, the nuclear interaction among hadrons of the fifth family members is very different than the ones so far observed [35,38].
d. The theory offers a new understanding of the second quantized fermion fields, as mentioned in **a.** and it is explained in Refs. [13,14], it also enables a new understanding of the second quantization of boson fields which is the main topics of this article [16,15], both in even dimensional spaces.
d.i. The Clifford odd "basis vectors" appear in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members. Their Hermitian conjugated partners appear in a separate group, Sect. 2.
d.ii. The Clifford even "basis vectors" appear in two groups, each with $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members with their Hermitian conjugated partners within the same group. One group of the Clifford even "basis vectors" transform, when applying algebraically on the Clifford odd "basis vector", this Clifford odd "basis vector" into other members of the same family. The other group of the Clifford even "basis vectors" transform, when being applied algebraically by the Clifford odd "basis vector", this Clifford odd "basis vector" into the same member of another family; in agreement with the action, Eq. (1).
d.iii. In odd dimensional spaces, $d=(2n+1)$, the properties of Clifford odd and Clifford even "basis vectors" differ essentially from their properties in even dimensional spaces, resembling the ghosts needed to make the contributions of the Feynman diagrams finite [18].

The theory seems very promising to offer a new insight into the second quantization of fermion and boson fields and to show the next step beyond the *standard model*.

The more work is put into the theory, the more phenomena the theory can explain.

Other references used a different approach by trying to make the next step with Clifford algebra to the second quantized fermion, which might also be a boson field [39,40].

Let us present a simple starting action of the *spin-charge-family* theory ([14] and the references therein) for massless fermions and anti-fermions which interact with massless gravitational fields only; with vielbeins (the gauge fields of momenta) and the two kinds of spin connection fields, the gauge fields of the two kinds of the Lorentz transformations in the internal space of fermions, of S^{ab} and $\tilde{S}^{ab}$, in $d=2(2n+1)$-dimensional space

$$
\begin{aligned}
\mathcal{A} &= \int \; d^d x \; E\, \frac{1}{2}\,(\bar{\psi}\,\gamma^a\, p_{0a}\psi) + h.c. + \\
&\quad \int \; d^d x \; E\,(\alpha\, R + \tilde{\alpha}\,\tilde{R})\,, \\
p_{0\alpha} &= p_\alpha - \frac{1}{2} S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}\,, \\
p_{0a} &= f^{\alpha}{}_a\, p_{0\alpha} + \frac{1}{2E}\,\{p_\alpha, E f^{\alpha}{}_a\}_-\,, \\
R &= \frac{1}{2}\,\{f^{\alpha[a} f^{\beta b]}\,(\omega_{ab\alpha,\beta} - \omega_{ca\alpha}\,\omega^{c}{}_{b\beta})\} + h.c.\,, \\
\tilde{R} &= \frac{1}{2}\,\{f^{\alpha[a} f^{\beta b]}\,(\tilde{\omega}_{ab\alpha,\beta} - \tilde{\omega}_{ca\alpha}\,\tilde{\omega}^{c}{}_{b\beta})\} + h.c.\,.
\end{aligned} \tag{1}
$$

Here[3] $f^{\alpha[a} f^{\beta b]} = f^{\alpha a} f^{\beta b} - f^{\alpha b} f^{\beta a}$. The vielbeins, f^a_α, and the two kinds of the spin connection fields, $\omega_{ab\alpha}$ (the gauge fields of S^{ab}) and $\tilde{\omega}_{ab\alpha}$ (the gauge fields of $\tilde{S}^{ab}$), manifest in $d = (3+1)$ as the known vector gauge fields and the scalar gauge fields taking care of masses of quarks and leptons and antiquarks and antileptons and of the weak boson fields [11,8,9,12].[4]

The action, Eq. (1), assumes two kinds of the spin connection gauge fields, due to two kinds of the operators: γ^a and $\tilde{\gamma}^a$. Let be pointed out that the description of the internal space of bosons with the Clifford even "basis vectors" offers as well two kinds of the Clifford even "basis vectors", as presented in **d.ii**.

In Sect. 2 the Grassmann and the Clifford algebras are explained, Subsect. 2.1, and creation and annihilation operators described as tensor products of the "basis vectors" offering an explanation of the internal spaces of fermion (by the Clifford odd algebra) and boson (by the Clifford even algebra) fields and the basis in ordinary space.

In Subsect. 2.2, the "basis vectors" are introduced and their properties presented in even and odd-dimensional spaces, Subsects. 2.2.1, Subsect. 2.2.2, respectively.

In Subsect. 2.3, the properties of the Clifford odd and even "basis vectors" are demonstrated in the toy model in $d = (5+1)$.

In Subsect. 2.4, the properties of the creation and annihilation operators for the second quantized fermion and boson fields in even dimensional spaces are described.

Sect. 3 presents what the reader could learn new from this article.

In App. B, the answers of the *spin-charge-family* theory to some of the open questions of the *standard model* are discussed.

In App. C, some useful formulas and relations are presented.

In App. D one irreducible representation (one family) of $SO(13,1)$, group, analysed with respect to $SO(3,1)$, $SU(2)_I$, $SU(2)_{II}$, $SU(3)$, and $U(1)$, representing "basis vectors" of quarks and leptons and antiquarks and antileptons is discussed.

App. A, suggested by the referee, illustrates on the simplest case $d = (3+1)$ (and $d = (1+1)$; which offers only one "family" of fermions, $d = (3+1)$ has two families) the properties of the Clifford odd and Clifford even "basis vectors" describing the internal spaces of fermion and boson fields, explaining in a pedagogical way in details their construction, manifestation of anticommutativity (in the fermion case) and commutativity (in the boson case) of the tensor product of the "basis vectors" and the basis in ordinary space-time.

The referee suggested also several footnotes.

[3] $f^\alpha{}_a$ are inverted vielbeins to $e^a{}_\alpha$ with the properties $e^a{}_\alpha f^\alpha{}_b = \delta^a{}_b$, $e^a{}_\alpha f^\beta{}_a = \delta^\beta_\alpha$, $E = \det(e^a{}_\alpha)$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau, ..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indexes from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.

[4] Since the multiplication with either γ^a's or $\tilde{\gamma}^a$'s changes the Clifford odd "basis vectors" into the Clifford even objects, and even "basis vectors" commute, the action for fermions can not include odd numbers of γ^a's or $\tilde{\gamma}^a$'s, what the simple starting action of Eq. (1) does not. In the starting action γ^a's and $\tilde{\gamma}^a$'s appear as $\gamma^0 \gamma^a \hat{p}_{0a}$ or as $\gamma^0 \gamma^c S^{ab} \omega_{abc}$ and as $\gamma^0 \gamma^c \tilde{S}^{ab} \tilde{\omega}_{abc}$.

2. Creation and annihilation operators for fermions and bosons in even and odd dimensional spaces

Refs. [6,10,13,8,14] describe the internal space of fermion fields by the superposition of odd products of γ^a in even dimensional spaces ($d = 2(2n+1)$, or $d = 4n$). In any even dimensional space there appear $2^{\frac{d}{2}-1}$ members of each irreducible representation of S^{ab}, each irreducible representation representing one of $2^{\frac{d}{2}-1}$ families, carrying quantum numbers determined by $\tilde{S}^{ab}$. Their Hermitian conjugated partners appear in a separate group (not reachable by either S^{ab} or $\tilde{S}^{ab}$). Since the tensor products, $*_T$, of these Clifford odd "basis vectors" and basis in ordinary momentum or coordinate space, applying on the vacuum state, fulfil the second quantization postulates for fermions [19–21], it is obvious that the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ anti-commuting Clifford odd "basis vectors", together with their Hermitian conjugated partners, transferring their anti-commutativity to creation and annihilation operators, explain the second quantization postulates of Dirac for fermions and their families [13].

There are, however, the same number of the Clifford even "basis vectors", which obviously commute, transferring their commutativity to tensor products, $*_T$, of the Clifford even "basis vectors" and basis in ordinary momentum or coordinate space.

We shall see in what follows that the Clifford even "basis vectors" appear in two groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. The members of each group have their Hermitian conjugated partners within the same group. As we shall see, one group transforms a particular family member of a Clifford odd "basis vector" into other members of the same family, keeping the family quantum number unchanged. The second group transforms a particular family member of a Clifford odd "basis vector" into the same member of another family [15]. We shall see that the Clifford even "basis vectors" of each of the two groups has, in even dimensional spaces, the properties of the gauge boson fields of the corresponding Clifford odd "basis vectors": One group with respect to S^{ab}, the other with respect to $\tilde{S}^{ab}$.

The properties of the Clifford odd and the Clifford even "basis vectors" in odd dimensional spaces, $d = (2n+1)$, differ essentially from their properties in even dimensional spaces, as we shall review Ref. [18] in Subsect. 2.2.2. Although anti-commuting, the Clifford odd "basis vectors" manifest properties of the Clifford even "basis vectors" in even dimensional spaces. And the Clifford even "basis vectors", although commuting, manifest properties of the Clifford odd "basis vectors" in even dimensional spaces.

2.1. Grassmann and Clifford algebras

This part is a short overview of several references, cited in Ref. ([14], Subsects. 3.2, 3.3), also appearing in Ref. [17,13,18].

The internal spaces of anti-commuting or commuting second quantized fields can be described by using either the Grassmann or the Clifford algebras [6,14].

In Grassmann d-dimensional space there are d anti-commuting (operators) θ^a, and d anti-commuting operators which are derivatives with respect to θ^a, $\frac{\partial}{\partial \theta_a}$,

$$\{\theta^a, \theta^b\}_+ = 0\,, \qquad \{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_+ = 0\,,$$

$$\{\theta_a, \frac{\partial}{\partial \theta_b}\}_+ = \delta_{ab}\,,\ (a,b) = (0,1,2,3,5,\cdots,d)\,. \tag{2}$$

Making a choice [12]

$$(\theta^a)^\dagger = \eta^{aa} \frac{\partial}{\partial \theta_a}\,, \quad \text{leads to} \quad (\frac{\partial}{\partial \theta_a})^\dagger = \eta^{aa} \theta^a\,, \tag{3}$$

with $\eta^{ab} = diag\{1,-1,-1,\cdots,-1\}$.

θ^a and $\frac{\partial}{\partial \theta_a}$ are, up to the sign, Hermitian conjugated to each other. The identity is the self adjoint member of the algebra. The choice for the following complex properties of θ^a

$$\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, ..., -\theta^{d-1}, \theta^d)\,, \tag{4}$$

correspondingly requires $\{\frac{\partial}{\partial \theta_a}\}^* = (\frac{\partial}{\partial \theta_0}, \frac{\partial}{\partial \theta_1}, -\frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3}, -\frac{\partial}{\partial \theta_5}, \frac{\partial}{\partial \theta_6}, ..., -\frac{\partial}{\partial \theta_{d-1}}, \frac{\partial}{\partial \theta_d})$.

There are 2^d superposition of products of θ^a, the Hermitian conjugated partners of which are the corresponding superposition of products of $\frac{\partial}{\partial \theta_a}$.

There exist two kinds of the Clifford algebra elements (operators), γ^a and $\tilde{\gamma}^a$, expressible with θ^a's and their conjugate momenta $p^{\theta a} = i\, \frac{\partial}{\partial \theta_a}$ [6], Eqs. (2), (3),

$$\begin{aligned}
&\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a})\,, \quad \tilde{\gamma}^a = i\,(\theta^a - \frac{\partial}{\partial \theta_a})\,,\\
&\theta^a = \frac{1}{2}\,(\gamma^a - i\tilde{\gamma}^a)\,, \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2}\,(\gamma^a + i\tilde{\gamma}^a)\,,
\end{aligned} \tag{5}$$

offering together $2 \cdot 2^d$ operators: 2^d are superposition of products of γ^a and 2^d of $\tilde{\gamma}^a$. It is easy to prove if taking into account Eqs. (3), (5), that they form two anti-commuting Clifford subalgebras, $\{\gamma^a, \tilde{\gamma}^b\}_+ = 0$, Refs. ([14] and references therein)

$$\begin{aligned}
&\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+\,,\\
&\{\gamma^a, \tilde{\gamma}^b\}_+ = 0\,, \quad (a,b) = (0,1,2,3,5,\cdots,d)\,,\\
&\quad (\gamma^a)^\dagger = \eta^{aa}\,\gamma^a\,, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa}\,\tilde{\gamma}^a\,.
\end{aligned} \tag{6}$$

While the Grassmann algebra offers the description of the "anti-commuting integer spin second quantized fields" and of the "commuting integer spin second quantized fields" [13,14], the Clifford algebras which are superposition of odd products of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the second quantized half integer spin fermion fields, which from the point of the subgroups of the $SO(d-1,1)$ group manifest spins and charges of fermions and antifermions in the fundamental representations of the group and subgroups, Table 4.

The superposition of even products of either γ^a's or $\tilde{\gamma}^a$'s offer the description of the commuting second quantized boson fields with integer spins (as we can see in [16,15] and shall see in this contribution) which from the point of the subgroups of the $SO(d-1,1)$ group manifest spins and charges in the adjoint representations of the group and subgroups.

The following *postulate*, which determines how does $\tilde{\gamma}^a$ operate on γ^a, reduces the two Clifford subalgebras, γ^a and $\tilde{\gamma}^a$, to one, to the one described by γ^a [10,6,9,12]

$$\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}\,|\psi_{oc}>\,, \tag{7}$$

with $(-)^B = -1$, if B is (a function of) odd products of γ^a's, otherwise $(-)^B = 1$ [10], the vacuum state $|\psi_{oc}>$ is defined in Eq. (15) of Subsect. 2.2.

After the postulate of Eq. (7) it follows:
a. The Clifford subalgebra described by $\tilde{\gamma}^a$'s looses its meaning for the description of the internal space of quantum fields.
b. The "basis vectors" which are superposition of odd or even products of γ^a's obey the postulates for the second quantized fields for fermions or bosons, respectively, Sect. 2.2.
c. It can be proven that the relations presented in Eq. (6) remain valid also after the postulate of Eq. (7). The proof is presented in Ref. ([14], App. I, Statement 3a).
d. Each irreducible representation of the Clifford odd "basis vectors" described by γ^a's are equipped by the quantum numbers of the Cartan subalgebra members of $\tilde{S}^{ab}$, chosen in Eq. (8), as follows

$$
\begin{aligned}
& S^{03}, S^{12}, S^{56}, \cdots, S^{d-1\,d}, \\
& \tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \cdots, \tilde{S}^{d-1\,d}, \\
& \mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab} = i\,(\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a})\,.
\end{aligned} \tag{8}
$$

After the postulate of Eq. (7) no vector space of $\tilde{\gamma}^a$'s needs to be taken into account for the description of the internal space of either fermions or bosons, in agreement with the observed properties of fermions and bosons. Also the Grassmann algebra is reduced to only one of the Clifford subalgebras. The operator $\tilde{\gamma}^a$ will from now on be used to describe the properties of fermion "basis vectors", determining by $\tilde{S}^{ab} = \frac{i}{4}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$ the "family" quantum numbers of the irreducible representations of the Lorentz group in internal space of fermions, S^{ab}, and the properties of bosons "basis vectors" determined by $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$. We shall see that while the fermion "basis vectors" appear in "families", the boson "basis vectors" have no "families" and manifest properties of the gauge fields of the corresponding fermion fields. In App. A the case of $d = (3+1)$ is discussed.

$\tilde{\gamma}^a$'s equip each irreducible representation of the Lorentz group (with the infinitesimal generators $S^{ab} = \frac{i}{4}\{\gamma^a, \gamma^b\}_-$) when applying on the Clifford odd "basis vectors" (which are superposition of odd products of $\gamma^{a's}$) with the family quantum numbers (determined by $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$).

Correspondingly the Clifford odd "basis vectors" (they are the superposition of odd products of γ^a's) form $2^{\frac{d}{2}-1}$ families, with the quantum number f, each family has $2^{\frac{d}{2}-1}$ members, m. They offer the description of the second quantized fermion fields.

The Clifford even "basis vectors" (they are the superposition of even products of γ^a's) have no families, as we shall see in what follows, but they do carry both quantum numbers, f and m, offering the description of the second quantized boson fields as the gauge fields of the second quantized fermion fields. The generators of the Lorentz transformations in the internal space of the Clifford even "basis vectors" are $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.

Properties of the Clifford odd and the Clifford even "basis vectors" are discussed in the following subsection.

2.2. *"Basis vectors" of fermions and bosons in even and odd dimensional spaces*

This subsection is a short overview of similar sections of several articles of the author, like [17, 15,18,13].

After the reduction of the two Clifford subalgebras to only one, Eq. (7), we only need to define "basis vectors" for the case that the internal space of second quantized fields is described by superposition of odd or even products γ^a's.[5]

Let us use the technique which makes "basis vectors" products of nilpotents and projectors [6, 10] which are eigenvectors of the (chosen) Cartan subalgebra members, Eq. (8), of the Lorentz algebra in the space of γ^a's, either in the case of the Clifford odd or in the case of the Clifford even products of γ^a's.

There are in even-dimensional spaces $\frac{d}{2}$ members of the Cartan subalgebra, Eq. (8). In odd-dimensional spaces there are $\frac{d-1}{2}$ members of the Cartan subalgebra.

One finds in even dimensional spaces for any of the $\frac{d}{2}$ Cartan subalgebra member, S^{ab} applying on a nilpotent $\overset{ab}{(k)}$ or on projector $\overset{ab}{[k]}$

$$
\begin{aligned}
&\overset{ab}{(k)} := \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b)\,, \quad (\overset{ab}{(k)})^2 = 0\,, \\
&\overset{ab}{[k]} := \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b)\,, \quad (\overset{ab}{[k]})^2 = \overset{ab}{[k]},
\end{aligned} \tag{9}
$$

the relations

$$
\begin{aligned}
&S^{ab} \overset{ab}{(k)} = \frac{k}{2} \overset{ab}{(k)}\,, \qquad \tilde{S}^{ab} \overset{ab}{(k)} = \frac{k}{2} \overset{ab}{(k)}\,, \\
&S^{ab} \overset{ab}{[k]} = \frac{k}{2} \overset{ab}{[k]}\,, \qquad \tilde{S}^{ab} \overset{ab}{[k]} = -\frac{k}{2} \overset{ab}{[k]}\,,
\end{aligned} \tag{10}
$$

with $k^2 = \eta^{aa}\eta^{bb}$,[6] demonstrating that the eigenvalues of S^{ab} on nilpotents and projectors expressed with γ^a differ from the eigenvalues of $\tilde{S}^{ab}$ on nilpotents and projectors expressed with γ^a, so that $\tilde{S}^{ab}$ can be used to equip each irreducible representation of S^{ab} with the "family" quantum number.[7]

We define in even d the "basis vectors" as algebraic, $*_A$, products of nilpotents and projectors so that each product is an eigenvector of all $\frac{d}{2}$ Cartan subalgebra members, Eq. (8). Fermion "basis vectors" are (algebraic, $*_A$) products of an odd number of nilpotents; each of them is the eigenvector of one of the Cartan subalgebra members, and the rest of the projectors; again is each projector the eigenvector of one of the Cartan subalgebra members. The boson "basis vectors" are (algebraic, $*_A$) products of an even number of nilpotents and the rest of the projectors. (In App. A, the reader can find concrete examples.)

It follows that the Clifford odd "basis vectors", which are the superposition of odd products of γ^a, must include an odd number of nilpotents, at least one, while the superposition of even products of γ^a, that is Clifford even "basis vectors", must include an even number of nilpotents or only projectors.

5 In Ref. [14], the reader can find in Subsects. (3.2.1 and 3.2.2) definitions for the "basis vectors" for the Grassmann and the two Clifford subalgebras, which are products of nilpotents and projectors chosen to be the eigenvectors of the corresponding Cartan subalgebra members of the Lorentz algebras presented in Eq. (8).

6 Let us prove one of the relations in Eq. (10): $S^{ab} \overset{ab}{(k)} = \frac{i}{2}\gamma^a\gamma^b\frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b) = \frac{1}{2^2}\{-i(\gamma^a)^2\gamma^b + i(\gamma^b)^2\gamma^a\frac{\eta^{aa}}{ik}\} = \frac{1}{2}\frac{\eta^{aa}\eta^{bb}}{k}\frac{1}{2}\{\gamma^a + \frac{k^2}{\eta^{bb}ik}\gamma^b\}$. For $k^2 = \eta^{aa}\eta^{bb}$ the first relation follows.

7 The reader can find the proof of Eq. (10) also in Ref. [14], App. (I).

We shall see that the Clifford odd "basis vectors" have properties appropriate to describe the internal space of the second quantized fermion fields while the Clifford even "basis vectors" have properties appropriate to describe the internal space of the second quantized boson fields.

Taking into account Eq. (6) one finds

$$\begin{aligned}
&\gamma^a \stackrel{ab}{(k)} = \eta^{aa} \stackrel{ab}{[-k]}, \quad \gamma^b \stackrel{ab}{(k)} = -ik \stackrel{ab}{[-k]}, \quad \gamma^a \stackrel{ab}{[k]} = \stackrel{ab}{(-k)}, \quad \gamma^b \stackrel{ab}{[k]} = -ik\eta^{aa} \stackrel{ab}{(-k)}, \\
&\tilde{\gamma^a} \stackrel{ab}{(k)} = -i\eta^{aa} \stackrel{ab}{[k]}, \quad \tilde{\gamma^b} \stackrel{ab}{(k)} = -k \stackrel{ab}{[k]}, \quad \tilde{\gamma^a} \stackrel{ab}{[k]} = \; i \stackrel{ab}{(k)}, \quad \tilde{\gamma^b} \stackrel{ab}{[k]} = -k\eta^{aa} \stackrel{ab}{(k)}, \\
&\stackrel{ab}{(k)}^{\dagger} = \eta^{aa} \stackrel{ab}{(-k)}, \quad (\stackrel{ab}{(k)})^2 = 0, \quad \stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa} \stackrel{ab}{[k]}, \\
&\stackrel{ab}{[k]}^{\dagger} = \stackrel{ab}{[k]}, \quad (\stackrel{ab}{[k]})^2 = \stackrel{ab}{[k]}, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0.
\end{aligned} \tag{11}$$

More relations are presented in App. C.

The relations in Eq. (11) demonstrate that the properties of "basis vectors" which include an odd number of nilpotents, differ essentially from the "basis vectors", which include an even number of nilpotents.

One namely recognizes:

i. Since the Hermitian conjugated partner of a nilpotent $\stackrel{ab}{(k)}^{\dagger}$ is $\eta^{aa} \stackrel{ab}{(-k)}$ and since neither S^{ab} nor $\tilde{S}^{ab}$ nor both can transform odd products of nilpotents to belong to one of the $2^{\frac{d}{2}-1}$ members of one of $2^{\frac{d}{2}-1}$ irreducible representations (families), the Hermitian conjugated partners of the Clifford odd "basis vectors" must belong to a different group of $2^{\frac{d}{2}-1}$ members of $2^{\frac{d}{2}-1}$ families.

Since S^{ac} transforms $\stackrel{ab}{(k)} *_A \stackrel{cd}{(k')}$ into $\stackrel{ab}{[-k]} *_A \stackrel{cd}{[-k']}$, while $\tilde{S}^{ac}$ transforms $\stackrel{ab}{(k)} *_A \stackrel{cd}{(k')}$ into $\stackrel{ab}{[k]} *_A \stackrel{cd}{[k']}$ it is obvious that the Hermitian conjugated partners of the Clifford even "basis vectors" must belong to the same group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Projectors are self-adjoint.

ii. Since odd products of γ^a anti-commute with another group of odd products of γ^a, the Clifford odd "basis vectors" anti-commute, manifesting in a tensor product, $*_T$, with the basis in ordinary space (together with the corresponding Hermitian conjugated partners) properties of the anti-commutation relations postulated by Dirac for the second quantized fermion fields.[8] The creation and annihilation operators, which include the internal space of fermions and bosons described by "basis vectors", the anti-commutativity or commutativity of which determine properties of the "basis vectors", fulfil the postulates of the second quantized fermion and boson fields. Basis of ordinary space commute as presented in Eq. (31). App. A discuses the creation and annihilation operators.

The Clifford even "basis vectors" correspondingly fulfil, in a tensor product, $*_T$, with the basis in ordinary space, the commutation relations for the second quantized boson fields.

iii. The Clifford odd "basis vectors" have all the eigenvalues of the Cartan subalgebra members equal to either $\pm\frac{1}{2}$ or to $\pm\frac{i}{2}$.

The Clifford even "basis vectors" have all the eigenvalues of the Cartan subalgebra members $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ equal to either ± 1 and zero or to $\pm i$ and zero.

---

[8] So far, we multiply nilpotents and projectors, or products of nilpotents and projectors forming "basis vectors", among themselves. With the tensor product, $*_T$, we include the basis in ordinary space.

In odd-dimensional spaces the "basis vectors" can not be products of only nilpotents and projections. As we shall see in Subsect. 2.2.2, half of "basis vectors" can be chosen as products of nilpotents and projectors, the rest can be obtained from the first half by the application of S^{0d} on the first half.

We shall demonstrate, shortly overviewing [18], that the second half of the "basis vectors" have unusual properties: The Clifford odd "basis vectors" have properties of the Clifford even "basis vectors", the Clifford even "basis vectors have properties of the Clifford odd "basis vectors".

2.2.1. *Clifford odd and even "basis vectors" in even d*

Let us define Clifford odd and even "basis vectors" as products of nilpotents and projectors in even-dimensional spaces.

a. *Clifford odd "basis vectors"*

This part overviews several papers with the same topic ([14,18] and references therein).

The Clifford odd "basis vectors" must be products of an odd number of nilpotents, and the rest, up to $\frac{d}{2}$, of projectors, each nilpotent and each projector must be the "eigenstate" of one of the members of the Cartan subalgebra, Eq. (8), correspondingly are the "basis vectors" eigenstates of all the members of the Lorentz algebra: S^{ab}'s determine $2^{\frac{d}{2}-1}$ members of one family, $\tilde{S}^{ab}$'s transform each member of one family to the same member of the rest of $2^{\frac{d}{2}-1}$ families.

Let us call the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, if it is the m^{th} membership of the family f. The Hermitian conjugated partner of $\hat{b}^{m\dagger}_{f}$ is called $\hat{b}^{m}_{f}$ $(= (\hat{b}^{m\dagger}_{f})^{\dagger}$.

Let us start in $d = 2(2n+1)$ with the "basis vector" $\hat{b}^{1\dagger}_{1}$ which is the product of only nilpotents, all the rest members belonging to the $f = 1$ family follow by the application of S^{01}, S^{03}, $\dots$, S^{0d}, S^{15}, $\dots$, S^{1d}, S^{5d} $\dots$, $S^{d-2\,d}$. They are presented on the left-hand side. Their Hermitian conjugated partners are presented on the right-hand side. The algebraic product mark $*_A$ among nilpotents and projectors is skipped.

$$
\begin{array}{ll}
 & d = 2(2n+1)\,, \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, & \hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{(+)}\,, & \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{(-)}\,, \\
\dots & \dots \\
\hat{b}^{2^{\frac{d}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}\,, & \hat{b}^{2^{\frac{d}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\stackrel{78}{[-]}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[-]}\,, \\
\dots\,, & \dots\,.
\end{array}
\tag{12}
$$

In $d = 4n$ the choice of the starting "basis vector" with maximal number of nilpotents must have one projector

$$
\begin{array}{ll}
 & d = 4n\,, \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\cdots\stackrel{d-1\,d}{[+]}\,, & \hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\cdots\stackrel{d-1\,d}{[+]} \\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-1\,d}{[+]}\,, & \hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-1\,d}{[+]}\,, \\
\dots\,, & \dots\,,
\end{array}
$$

$$\hat{b}_1^{2^{\frac{d}{2}-1}\dagger} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]}\,, \qquad \hat{b}_1^{2^{\frac{d}{2}-1}} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(-)}\cdots\stackrel{d-3\,d-2}{[-]}\stackrel{d-1\,d}{[+]}\,,$$
$$\cdots\,, \qquad \cdots\,. \tag{13}$$

The Hermitian conjugated partners of the Clifford odd "basis vectors" $\hat{b}_1^{m\dagger}$, presented in Eq. (13) on the right-hand side, follow if all nilpotents $\stackrel{ab}{(k)}$ of $\hat{b}_1^{m\dagger}$ are transformed into $\eta^{aa}\stackrel{ab}{(-k)}$.

For either $d = 2(2n+1)$ or for $d = 4n$ all the $2^{\frac{d}{2}-1}$ families follow by applying $\tilde{S}^{ab}$'s on all the members of the starting family. (Or one can find the starting $\hat{b}_f^{1\dagger}$ for all families f and then generate all the members $\hat{b}_f^m$ from $\hat{b}_f^{1\dagger}$ by the application of S^{ab} on the starting member.)

It is not difficult to see that all the "basis vectors" within any family, as well as the "basis vectors" among families, are orthogonal; that is, their algebraic product is zero. The same is true within their Hermitian conjugated partners. Both can be proved by the algebraic multiplication using Eqs. (11), (47).

$$\hat{b}_f^{m\dagger} *_A \hat{b}_{f`}^{m`\dagger} = 0\,, \quad \hat{b}_f^{m} *_A \hat{b}_{f`}^{m`} = 0\,, \quad \forall m, m', f, f`\,. \tag{14}$$

When we choose the vacuum state equal to

$$|\psi_{oc}> = \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}_f^{m}{}_{*_A}\hat{b}_f^{m\dagger}\,|\,1>\,, \tag{15}$$

for one of members m, which can be anyone of the odd irreducible representations f it follows that the Clifford odd "basis vectors" obey the relations

$$\begin{aligned}
\hat{b}_f^{m}{}_{*_A}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\hat{b}_f^{m\dagger}{}_{*_A}|\psi_{oc}> &= |\psi_f^m>\,,\\
\{\hat{b}_f^{m}, \hat{b}_{f`}^{m'}\}_{*_A+}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\{\hat{b}_f^{m\dagger}, \hat{b}_{f`}^{m'\dagger}\}_{*_A+}|\psi_{oc}> &= 0.\,|\psi_{oc}>\,,\\
\{\hat{b}_f^{m}, \hat{b}_{f`}^{m'\dagger}\}_{*_A+}|\psi_{oc}> &= \delta^{mm'}\,\delta_{ff`}|\psi_{oc}>\,,
\end{aligned} \tag{16}$$

while the normalization $<\psi_{oc}|\hat{b}_f^{m\dagger} *_A \hat{b}_f^{m\dagger} *_A |\psi_{oc}> = 1$ is used and the anti-commutation relation mean $\{\hat{b}_f^{m\dagger}, \hat{b}_{f`}^{m'\dagger}\}_{*_A+} = \hat{b}_f^{m\dagger} *_A \hat{b}_{f`}^{m'\dagger} + \hat{b}_{f`}^{m'\dagger} *_A \hat{b}_f^{m\dagger}$.

If we write the creation and annihilation operators for fermions as the tensor, $*_T$, products of "basis vectors" and the basis in ordinary space, the creation and annihilation operators fulfil Dirac's anti-commutation postulates since the "basis vectors" transfer their anti-commutativity to creation and annihilation operators; the ordinary basis namely commute as presented in Eqs. (31), (32). Describing the internal space of fermions with the Clifford odd "basis vectors", makes creation operators fulfilling the Dirac postulates for the second quantized fermion fields: No postulates are needed. The creation and annihilation operators for fermions and bosons are discussed in App. A, in the part with the title "Creation and annihilation operators".

It turns out, therefore, that not only the Clifford odd "basis vectors" offer the description of the internal space of fermions, they explain the second quantization postulates for fermions as well.

Table 1, presented in Subsect. 2.3, illustrates the properties of the Clifford odd "basis vectors" on the case of $d = (5+1)$.

b. *Clifford even "basis vectors"*

This part proves that the Clifford even "basis vectors" are in even-dimensional spaces offering the description of the internal spaces of boson fields — the gauge fields of the corresponding Clifford odd "basis vectors": It is a new recognition, offering a new understanding of the second quantized fermion and **boson** fields [15].

The Clifford even "basis vectors" must be products of an even number of nilpotents and the rest, up to $\frac{d}{2}$, of projectors; each nilpotent and each projector is chosen to be the "eigenstate" of one of the members of the Cartan subalgebra of the Lorentz algebra, $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (8). Correspondingly the "basis vectors" are the eigenstates of all the members of the Cartan subalgebra of the Lorentz algebra.

The Clifford even "basis vectors" appear in two groups, each group has $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. The members of one group can not be reached from the members of another group by either S^{ab}'s or $\tilde{S}^{ab}$'s or both.

S^{ab} and $\tilde{S}^{ab}$ generate from the starting "basis vector" of each group all the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. Each group contains the Hermitian conjugated partner of any member; $2^{\frac{d}{2}-1}$ members of each group are products of only (self adjoint) projectors.

Let us call the Clifford even "basis vectors" ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, where $i = (I, II)$ denotes the two groups of Clifford even "basis vectors", while m and f determine membership of "basis vectors" in any of the two groups, I or II.

$$
\begin{aligned}
& \qquad\qquad\qquad\qquad d = 2(2n+1) \\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} &= \overset{03}{(+i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{[+]}\,, &\quad {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} &= \overset{03}{(-i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{[+]}\,, \\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} &= \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-1\,d}{[+]}\,, &\quad {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} &= \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{(+)}\cdots\overset{d-1\,d}{[+]}\,, \\
{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} &= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{(-)}\,, &\quad {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} &= \overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{(-)}\,, \\
& \qquad\qquad\qquad\qquad\qquad \cdots & & \cdots \\
& \qquad\qquad\qquad\qquad d = 4n \\
{}^{I}\hat{\mathcal{A}}^{1\dagger}_{1} &= \overset{03}{(+i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{(+)}\,, &\quad {}^{II}\hat{\mathcal{A}}^{1\dagger}_{1} &= \overset{03}{(-i)}\overset{12}{(+)}\cdots\overset{d-1\,d}{(+)}\,, \\
{}^{I}\hat{\mathcal{A}}^{2\dagger}_{1} &= \overset{03}{[-i]}\overset{12}{[-i]}\overset{56}{(+)}\cdots\overset{d-1\,d}{(+)}\,, &\quad {}^{II}\hat{\mathcal{A}}^{2\dagger}_{1} &= \overset{03}{[+i]}\overset{12}{[-i]}\overset{56}{(+)}\cdots\overset{d-1\,d}{(+)}\,, \\
{}^{I}\hat{\mathcal{A}}^{3\dagger}_{1} &= \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{[-]}\,, &\quad {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1} &= \overset{03}{(-i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{[-]}\overset{d-1\,d}{[-]} \\
& \qquad\qquad\qquad\qquad\qquad \cdots & & \cdots
\end{aligned}
\tag{17}
$$

There are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and there are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Clifford even "basis vectors" of the kind ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$.

Table 1, presented in Subsect. 2.3, illustrates properties of the Clifford odd and Clifford even "basis vectors" on the case of $d = (5+1)$. Looking at this case it is easy to evaluate properties of either even or odd "basis vectors". We shall discuss in this subsection the general case by carefully inspecting properties of both kinds of "basis vectors".

The Clifford even "basis vectors" belonging to two different groups are orthogonal due to the fact that they differ in the sign of one nilpotent or one projector, or the algebraic product of a member of one group with a member of another group gives zero according to the first two lines of Eq. (47): $\overset{ab}{(k)}\overset{ab}{[k]}=0$, $\overset{ab}{[k]}\overset{ab}{(-k)}=0$, $\overset{ab}{[k]}\overset{ab}{[-k]}=0$.

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} = 0 = {}^{II}\hat{\mathcal{A}}^{m\dagger}{}_{\cdot f} *_A {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} \,. \tag{18}
$$

The members of each of these two groups have the property

$$
{}^{i}\hat{\mathcal{A}}^{m\dagger}_{f} *_A {}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f‘} \to \begin{cases} {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f‘}\,, i=(I,II) \\ \qquad\qquad \text{or zero}\,. \end{cases} \tag{19}
$$

For a chosen $(m, f, f‘)$ there is only one m' (out of $2^{\frac{d}{2}-1}$) which gives nonzero contribution.

Two “basis vectors”, ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{i}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, the algebraic product, $*_A$, of which gives non zero contribution, “scatter” into the third one ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f‘}$, for $i=(I,II)$.

Let us treat a particular case in $d=2(2n+1)$-dimensional internal space, like: ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} = \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{(+)}\overset{d-1\,d}{[+]} *_A \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(-)}\cdots\overset{d-3\,d-2}{(-)}\overset{d-1\,d}{[+]} \to \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[+]}\cdots\overset{d-3\,d-2}{[+]}\overset{d-1\,d}{[+]}$, what follows if the first two lines of Eq. (47) are taken into account. The eigenvalues of the Cartan subalgebra members of $\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{(+)}\overset{d-1\,d}{[+]}$ are $(i,1,1,1,\dots,1,0)$, of $\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(-)}\cdots\overset{d-3\,d-2}{(-)}\overset{d-1\,d}{[+]}$ are $(0,0,-1,-1,\dots,-1,0)$, and of $\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[+]}\cdots\overset{d-3\,d-2}{[+]}\overset{d-1\,d}{[+]}$ are $(i,1,0,0,\dots,0,0)$. The sum of the Cartan subalgebra eigenvalues of the two scattered Clifford even “basis vectors” leads to the eigenvalues $(i,1,0,0,\dots,0,0)$ of the third Clifford even “basis vector”.

It remains to evaluate the algebraic application, $*_A$, of the Clifford even “basis vectors” ${}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd “basis vectors” $\hat b^{m'\dagger}_{f‘}$. One finds, taking into account Eq. (47), for ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$

$$
{}^{I}\hat{\mathcal{A}}^{m\dagger}_{f} *_A \hat b^{m'\dagger}_{f‘} \to \begin{cases} \qquad \hat b^{m\dagger}_{f‘}\,, \\ \text{or zero}\,. \end{cases} \tag{20}
$$

For each ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members of the Clifford odd “basis vectors” (describing the internal space of fermion fields) $2^{\frac{d}{2}-1}$ members, $\hat b^{m'\dagger}_{f‘}$, fulfilling the relation of Eq. (20). All the rest $(2^{\frac{d}{2}-1}\times(2^{\frac{d}{2}-1}-1))$ Clifford odd “basis vectors” give zero contributions. Or equivalently, there are $2^{\frac{d}{2}-1}$ pairs of quantum numbers (f, m') for which $\hat b^{m\dagger}_{f‘}\neq 0$.

Taking into account Eq. (47) one finds

$$
\hat b^{m\dagger}_{f} *_A {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f‘} = 0\,, \quad \forall (m, m‘, f, f‘)\,. \tag{21}
$$

Let us treat a particular case in $d=2(2n+1)$-dimensional space: ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}(\equiv \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\cdots\overset{d-3\,d-2}{(+)}\overset{d-1\,d}{[+]}) *_A \hat b^{m'\dagger}_{f‘}(\equiv \overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(-)}\cdots\overset{d-3\,d-2}{(-)}\overset{d-1\,d}{(+)}) \to \hat b^{m\dagger}_{f‘}(\equiv \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[+]}\cdots\overset{d-3\,d-2}{[+]}\overset{d-1\,d}{(+)})$. The $\mathbf{S}^{ab}$ (meaning $\mathbf{S}^{03},\mathbf{S}^{12},\mathbf{S}^{56},\dots\mathbf{S}^{d-1\,d}$) say for the above case that the boson field with the quantum numbers $(i,1,1,\dots,1,0)$ when “scattering” on the fermion field with the Cartan subalgebra quantum numbers $(S^{03},S^{1,2},S^{56}\dots S^{d-3\,d-2},S^{d-1\,d})=(-\frac{i}{2},-\frac{1}{2},-\frac{1}{2},\dots,-\frac{1}{2},\frac{1}{2})$, and the family quantum numbers $(-\frac{i}{2},-\frac{1}{2},-\frac{1}{2},\dots,-\frac{1}{2},\frac{1}{2})$ transfers to the fermion field its quantum numbers

$(i, 1, 1, \ldots, 1, 0)$, transforming fermion family members quantum numbers to $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$, leaving family quantum numbers unchanged.

Eqs. (20), (21) demonstrates that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, applying on $\hat{b}^{m'\dagger}_{f`}$, transforms the Clifford odd "basis vector" into another Clifford odd "basis vector" of the same family, transferring to the Clifford odd "basis vector" integer spins, or gives zero.

For "scattering" the Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f`}$ it follows

$$
{}^{II}\hat{\mathcal{A}}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f`} = 0\,, \;\; \forall (m, m', f, f`)\,, \tag{22}
$$

while we get

$$
\hat{b}^{m\dagger}_{f} *_A {}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`} \rightarrow \left\{ \begin{array}{c} \hat{b}^{m\dagger}_{f``}\,, \\ \text{or zero}\,. \end{array} \right. \tag{23}
$$

For each $\hat{b}^{m\dagger}_{f}$ there are among $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford even "basis vectors" (describing the internal space of boson fields), ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$, $2^{\frac{d}{2}-1}$ members (with appropriate $f`$ and m') fulfilling the relation of Eq. (23) while $f``$ runs over $(1 - 2^{\frac{d}{2}-1})$.

All the rest $(2^{\frac{d}{2}-1} \times (2^{\frac{d}{2}-1} - 1))$ Clifford even "basis vectors" give zero contributions.

Or equivalently, there are $2^{\frac{d}{2}-1}$ pairs of quantum numbers (f', m') for which $\hat{b}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f`}$ give non zero contribution.

Let us treat a particular case in $d = 2(2n+1)$-dimensional space:
$\hat{b}^{m\dagger}_{f}(\equiv \overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(-)} \cdots \overset{d-3\,d-2}{(-)}\;\overset{d-1\,d}{(+)}) *_A {}^{II}\hat{\mathcal{A}}^{m`\dagger}_{f`}(\equiv \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} \cdots \overset{d-3\,d-2}{(+)}\;\overset{d-1\,d}{[-]}) \rightarrow \hat{b}^{m\dagger}_{f``}(\equiv \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]} \cdots \overset{d-3\,d-2}{[-]}\;\overset{d-1\,d}{(+)})$ When the fermion field with the Cartan subalgebra family members quantum numbers $(S^{03}, S^{12}, S^{56} \ldots\; S^{d-3\,d-2}, S^{d-1\,d}) = (-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})$ and family quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56} \ldots \tilde{S}^{d-3\,d-2}, \tilde{S}^{d-1\,d})$ $(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2})$ "absorbs" a boson field with the Cartan subalgebra quantum numbers $\mathbf{S}^{ab}$ (meaning $\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \ldots \mathbf{S}^{d-1\,d}$) equal to $(i, 1, 1, \ldots, 1, 0)$, the fermion field changes the family quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56} \ldots \tilde{S}^{d-3\,d-2}, \tilde{S}^{d-1\,d})$ to $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$, keeping family members quantum numbers unchanged.

Eqs. (22), (23) demonstrate that ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, "absorbed" by $\hat{b}^{m\dagger}_{f}$, transforms the Clifford odd "basis vector" into the Clifford odd "basis vector" of the same family member and of another family, or gives zero.

The Clifford even "basis vectors" offer the description of the internal space of the gauge fields of the corresponding fermion fields.

While the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, offer the description of the internal space of the second quantized anti-commuting fermion fields, appearing in families, the Clifford even "basis vectors", ${}^{I,II}\hat{\mathcal{A}}^{m\dagger}_{f}$, offer the description of the internal space of the second quantized commuting boson fields, having no families and appearing in two groups. One of the two groups, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, transferring their integer quantum numbers to the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, changes the family members quantum numbers leaving the family quantum numbers unchanged. The second group, transferring their integer quantum numbers to the Clifford odd "basis vector", changes the family quantum numbers leaving the family members quantum numbers unchanged.

Both groups of Clifford even "basis vectors" manifest as the gauge fields of the corresponding fermion fields: One concerning the family members quantum numbers, the other concerning the family quantum numbers.

We shall discus properties of the Clifford even and odd "basis vectors" for $d = (5+1)$-dimensional internal spaces in Subsect. 2.3 in more details.

2.2.2. Clifford odd and even "basis vectors" in d odd

Let us shortly overview properties of the fermion and boson "basis vectors" in odd dimensional spaces, as presented in Ref. [18], Subsect. 2.2.

In even dimensional spaces the Clifford odd "basis vectors" fulfil the postulates for the second quantized fermion fields, Eq. (16), and the Clifford even "basis vectors" have the properties of the internal spaces of their corresponding gauge fields, Eqs. (19), (20), (23). In odd dimensional spaces, the Clifford odd and even "basis vectors" have unusual properties resembling properties of the internal spaces of the Faddeev–Popov ghosts, as we described in [18].

In $d = (2n+1)$-dimensional cases, $n = 1, 2, \ldots$, half of the "basis vectors", $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$, can be taken from the $2n$-dimensional part of space, presented in Eqs. (12), (13), (17), (19).

The rest of the "basis vectors" in odd dimensional spaces, $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$, follow if $S^{0\,2n+1}$ is applied on these half of the "basis vectors". Since $S^{0\,2n+1}$ are Clifford even operators, they do not change the oddness or evenness of the "basis vectors".

For the Clifford odd "basis vectors", the $2^{\frac{d-1}{2}-1}$ members appearing in $2^{\frac{d-1}{2}-1}$ families and representing the part which is the same as in even, $d = 2n$, dimensional space are present on the left-hand side of Eq. (24), the part obtained by applying $S^{0\,2n+1}$ on the one of the left-hand side is presented on the right hand side. Below the "basis vectors" and their Hermitian conjugated partners are presented.

$$
\begin{array}{llll}
 & d = 2(2n+1)+1 & & \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{(+)}\,, & & \hat{b}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{(+)}\;\gamma^{d}\,, & \\
\qquad\cdots & & \cdots & \\
\hat{b}^{2^{\frac{d-1}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{[-]}\,, & & \hat{b}^{2^{\frac{d-1}{2}-1}\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{[-]}\;\gamma^{d}\,, & \\
\qquad\cdots & & \cdots\,, & \\
 & & \cdots\,, & \\
\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(-)}\cdots\stackrel{d-2\,d-1}{(-)}\,, & & \hat{b}^{1}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[+i]}\stackrel{12}{(-)}\stackrel{56}{(-)}\cdots\stackrel{d-2\,d-1}{(-)}\;\gamma^{d}\,, & \\
\qquad\cdots & & \cdots\,. &
\end{array}
\tag{24}
$$

The application of S^{0d} or $\tilde{S}^{0d}$ on the left-hand side of the "basis vectors" (and the Hermitian conjugated partners of both) generate the whole set of $2 \times 2^{d-2}$ members of the Clifford odd "basis vectors" and their Hermitian conjugated partners in $d = (2n+1)$-dimensional space appearing on the left-hand side and the right-hand sides of Eq. (24).

It is not difficult to see that $\hat{b}^{m\dagger}_{2^{\frac{d-1}{2}-1}+k}$ and $\hat{b}^{m'}_{2^{\frac{d-1}{2}-1}+k'}$ on the right-hand side of Eq. (24) obtain properties of the two groups (they are orthogonal to each other; the algebraic products, $*_A$, of a member from one group, and any member of another group give zero) with the Hermitian conjugated partners within the same group; they have properties of the Clifford even "basis vectors" from the point of view of the Hermiticity property: The operators γ^a are up to a constant the self-adjoint operators, while S^{0d} transform one nilpotent into a projector.

S^{ab} do not change the Clifford oddness of $\hat{b}^{m\dagger}_f$, and $\hat{b}^m_f; \hat{b}^{m\dagger}_f$ remain to be Clifford odd objects, however, with the properties of boson fields.

Let us find the Clifford even "basis vectors" in odd dimensional space $d = 2(2n+1)+1$.

$$
\begin{aligned}
& d = 2(2n+1)+1 \\
& {}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{[+]}\,, \quad {}^{I}\mathcal{A}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[-i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{[+]}\;\gamma^d\,, \\
& \qquad\cdots \quad \cdots \\
& {}^{I}\mathcal{A}^{2^{\frac{d-1}{2}-1}\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-2\,d-1}{[+]}\,, \quad {}^{I}\mathcal{A}^{2^{\frac{d-1}{2}-1}\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{(+i)}\stackrel{12}{[-]}\stackrel{56}{[-]}\cdots\stackrel{d-2\,d-1}{[+]}\;\gamma^d\,, \\
& \qquad\cdots \quad \cdots\,, \\
& \qquad\cdots \quad \cdots \\
& {}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{[+]}\,, \quad {}^{II}\mathcal{A}^{1\dagger}_{2^{\frac{d-1}{2}-1}+1} = \stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(+)}\cdots\stackrel{d-2\,d-1}{[+]}\;\gamma^d\,, \\
& \qquad\cdots \quad \cdots\,.
\end{aligned}
\tag{25}
$$

The right hand side of Eq. (24), although anti-commuting, is resembling the properties of the Clifford even "basis vectors" on the left hand side of Eq. (25), while the right-hand side of Eq. (25), although commuting, resembles the properties of the Clifford odd "basis vectors", from the left hand side of Eq. (24): γ^a are up to a constant the self adjoint operators, while S^{0d} transform one nilpotent into a projector (or one projector into a nilpotent). However, S^{ab} do not change Clifford evenness of ${}^{I}\mathcal{A}^{m\dagger}_{f}$, $i=(I, II)$.

For illustration let me copy the special case for $d=(4+1)$ from Subsect. 3.2.2. of Ref. [18].

$$
\begin{aligned}
& d = 4+1 \\
& \text{Clifford odd} \\
& \hat{b}^{1\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,,\; \hat{b}^{1\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,,\; \hat{b}^{1\dagger}_3 = \stackrel{03}{[-i]}\stackrel{12}{[+]}\,\gamma^5\,,\; \hat{b}^{1\dagger}_4 = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,\gamma^5\,, \\
& \hat{b}^{2\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,,\; \hat{b}^{2\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,,\; \hat{b}^{2\dagger}_3 = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,\gamma^5\,,\; \hat{b}^{2\dagger}_4 = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,\gamma^5\,, \\
& \hat{b}^{1}_1 = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,,\; \hat{b}^{1}_2 = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,,\; \hat{b}^{1}_3 = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,\gamma^5\,,\; \hat{b}^{1}_4 = \stackrel{03}{(-i)}\stackrel{12}{(-)}\,\gamma^5\,, \\
& \hat{b}^{2}_1 = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,,\; \hat{b}^{2}_2 = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,,\; \hat{b}^{2}_3 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,\gamma^5\,,\; \hat{b}^{2}_4 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,\gamma^5\,, \\
& \\
& \text{Clifford even} \\
& {}^{I}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[+i]}\stackrel{12}{[+]}\,,\; {}^{I}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(+i)}\stackrel{12}{(+)}\,,\; {}^{I}\mathcal{A}^{1}_3 = \stackrel{03}{(-i)}\stackrel{12}{[+]}\,\gamma^5\,,\; {}^{I}\mathcal{A}^{1}_4 = \stackrel{03}{[-i]}\stackrel{12}{(+)}\,\gamma^5\,, \\
& {}^{I}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(-i)}\stackrel{12}{(-i)}\,,\; {}^{I}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,,\; {}^{I}\mathcal{A}^{2}_3 = \stackrel{03}{[+i]}\stackrel{12}{(-)}\,\gamma^5\,,\; {}^{I}\mathcal{A}^{2}_4 = \stackrel{03}{(+i)}\stackrel{12}{[-]}\,\gamma^5\,, \\
& \\
& {}^{II}\mathcal{A}^{1\dagger}_1 = \stackrel{03}{[-i]}\stackrel{12}{[+]}\,,\; {}^{II}\mathcal{A}^{1\dagger}_2 = \stackrel{03}{(-i)}\stackrel{12}{(+)}\,,\; {}^{II}\mathcal{A}^{1\dagger}_3 = \stackrel{03}{(+i)}\stackrel{12}{[+]}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{1\dagger}_4 = \stackrel{03}{[+i]}\stackrel{12}{(+)}\,\gamma^5\,, \\
& {}^{II}\mathcal{A}^{2\dagger}_1 = \stackrel{03}{(+i)}\stackrel{12}{(-)}\,,\; {}^{II}\mathcal{A}^{2\dagger}_2 = \stackrel{03}{[+i]}\stackrel{12}{[-]}\,,\; {}^{II}\mathcal{A}^{2\dagger}_3 = \stackrel{03}{[-i]}\stackrel{12}{(-)}\,\gamma^5\,,\; {}^{II}\mathcal{A}^{2\dagger}_4 = \stackrel{03}{(-i)}\stackrel{12}{[-]}\,\gamma^5\,.
\end{aligned}
\tag{26}
$$

It can clearly be seen that the left-hand side of the Clifford odd "basis vectors" and the right-hand side of the Clifford even "basis vectors", although the former are the Clifford odd objects and the latter are Clifford even objects, have similar properties [18].

2.3. Example demonstrating properties of Clifford odd and even "basis vectors" for $d = (5 + 1)$

Subsect. 2.3 demonstrates the properties of the Clifford odd and even "basis vectors" in the special case when $d = (5 + 1)$ to clear up the relations of the Clifford odd and even "basis vectors" to fermion and boson fields, respectively.

Table 1 presents the $64\,(= 2^{d=6})$ "eigenvectors" of the Cartan subalgebra members of the Lorentz algebra, S^{ab} and $\mathcal{S}^{ab}$, Eq. (8).

The Clifford odd "basis vectors" — they appear in $4\,(= 2^{\frac{d=6}{2}-1})$ families, each family has 4 members — are products of an odd number of nilpotents, either of three or one. They appear in the group named *odd I* $\hat{b}^{m\dagger}_f$. Their Hermitian conjugated partners appear in the second group named *odd II* $\hat{b}^{m}_f$. Within each of these two groups the members are mutually orthogonal (which can be checked by using Eq. (47)); $\hat{b}^{m\dagger}_f *_A \hat{b}^{m'\dagger}_{f`} = 0$ for all $(m, m', f, f`)$. Equivalently, $\hat{b}^{m}_f *_A \hat{b}^{m'}_{f`} = 0$ for all $(m, m', f, f`)$. The "basis vectors" and their Hermitian conjugated partners are normalized as

$$< \psi_{oc}|\hat{b}^{m}_f *_A \hat{b}^{m'\dagger}_{f`}|\psi_{oc} >= \delta^{mm'}\delta_{ff`}\,, \tag{27}$$

since the vacuum state $|\psi_{oc} >= \frac{1}{\sqrt{2^{\frac{d=6}{2}-1}}}\,(\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]} + \overset{03}{[-i]}\overset{12}{[+]}\overset{56}{[+]} + \overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[+]} + \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[-]})$ is normalized to one: $< \psi_{oc}|\psi_{oc} >= 1$.

The more extended overview of the properties of the Clifford odd "basis vectors" and their Hermitian conjugated partners for the case $d = (5 + 1)$ can be found in Ref. [14].

The Clifford even "basis vectors" are products of an even number of nilpotents — of either two or none in this case. They are presented in Table 1 in two groups, each with $16\,(= 2^{\frac{d=6}{2}-1} \times 2^{\frac{d=6}{2}-1})$ members, as *even I* $\mathcal{A}^{m\dagger}_f$ and *even II* $\mathcal{A}^{m\dagger}_f$. One can easily check, using Eq. (47), that the algebraic product ${}^{I}\mathcal{A}^{m\dagger}_f *_A {}^{II}\mathcal{A}^{m'\dagger}_{f`} = 0 = {}^{II}\mathcal{A}^{m\dagger}_f *_A {}^{I}\mathcal{A}^{m'\dagger}_{f`}, \forall\,(m, m', f. f`)$, Eq. (18). An overview of the Clifford even "basis vectors" and their Hermitian conjugated partners for the case $d = (5 + 1)$ can be found in Ref. [15].

While the Clifford odd "basis vectors" are (chosen to be) left handed, $\Gamma^{(5+1)} = -1$, their Hermitian conjugated partners have opposite handedness, Eq. (45) in App. C.[9]

While the Clifford odd "basis vectors" have half integer eigenvalues of the Cartan subalgebra members, Eq. (8), that is of S^{03}, S^{12}, S^{56} in this particular case of $d = (5 + 1)$, the Clifford even "basis vectors" have integer spins, obtained by $\mathcal{S}^{03} = S^{03} + \tilde{S}^{03}$, $\mathcal{S}^{12} = S^{12} + \tilde{S}^{12}$, $\mathcal{S}^{56} = S^{56} + \tilde{S}^{56}$.

Let us check what does the algebraic application, $*_A$, of ${}^{I}\hat{\mathcal{A}}^{m=1\dagger}_{f=4}$, for example, presented in Table 1 in the first line of the fourth column of *even I*, do on the Clifford odd "basis vector"

---

[9] Let us check the handedness of the chosen representation: $\Gamma^{5+1}\hat{b}^{1\dagger}_1 (\equiv \overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}) = \sqrt{(-1)^5}\, i^3 (\frac{2}{i})^3 S^{03} S^{12} S^{56} (\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}) = \frac{i^4 2^3}{i^3}\frac{i}{2}\frac{1}{2}\frac{1}{2}(\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}) = -1(\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{[+]}))$.

Table 1
$2^d = 64$ "eigenvectors" of the Cartan subalgebra of the Clifford odd and even algebras — the superposition of odd and even products of γ^a's — in $d = (5+1)$-dimensional space are presented, divided into four groups. The first group, *odd I*, is chosen to represent "basis vectors", named $\hat{b}^{m\dagger}_f$, appearing in $2^{\frac{d}{2}-1} = 4$ "families" ($f = 1, 2, 3, 4$), each "family" with $2^{\frac{d}{2}-1} = 4$ "family" members ($m = 1, 2, 3, 4$). The second group, *odd II*, contains Hermitian conjugated partners of the first group for each family separately, $\hat{b}^m_f = (\hat{b}^{m\dagger}_f)^\dagger$. Either *odd I* or *odd II* are products of an odd number of nilpotents (one or three) and projectors (two or none). The "family" quantum numbers of $\hat{b}^{m\dagger}_f$, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, are for the first *odd I* group appearing above each "family", the quantum numbers of the family members (S^{03}, S^{12}, S^{56}) are written in the last three columns. For the Hermitian conjugated partners of *odd I*, presented in the group *odd II*, the quantum numbers (S^{03}, S^{12}, S^{56}) are presented above each group of the Hermitian conjugated partners, the last three columns tell eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$. The two groups with the even number of γ^a's, *even I* and *even II*, each group has their Hermitian conjugated partners within its group, have the quantum numbers f, that is the eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, written above column of four members, the quantum numbers of the members, (S^{03}, S^{12}, S^{56}), are written in the last three columns. To find the quantum numbers of $(\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56})$ one has to take into account that $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$.

"*basis vectors*" $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	m $\rightarrow$	$f=1$ $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$	$f=2$ $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$	$f=3$ $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$	$f=4$ $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$	S^{03}	S^{12}	S^{56}
		03 12 56	03 12 56	03 12 56	03 12 56			
odd I $\hat{b}^{m\dagger}_f$	1	$(+i)[+][+]$	$[+i][+](+)$	$[+i](+)[+]$	$(+i)(+)(+)$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$[-i](-)[+]$	$(-i)(-)(+)$	$(-i)[-][+]$	$[-i][-](+)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$[-i][+](-)$	$(-i)[+][-]$	$(-i)(+)(-)$	$[-i](+)[-]$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$(+i)(-)(-)$	$[+i](-)[-]$	$[+i][-](-)$	$(+i)[-][-]$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
(S^{03}, S^{12}, S^{56})	$\rightarrow$	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$
odd II $\hat{b}^m_f$	1	$(-i)[+][+]$	$[+i][+](-)$	$[+i](-)[+]$	$(-i)(-)(-)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	$[-i](+)[+]$	$(+i)(+)(-)$	$(+i)[-][+]$	$[-i][-](-)$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	$[-i][+](+)$	$(+i)[+][-]$	$(+i)(-)(+)$	$[-i](-)[-]$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	4	$(-i)(+)(+)$	$[+i](+)[-]$	$[+i][-](+)$	$(-i)[-][-]$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	$\rightarrow$	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even I ${}^{I}\mathcal{A}^m_f$	1	$[+i](+)(+)$	$(+i)[+](+)$	$[+i][+][+]$	$(+i)(+)[+]$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$(-i)[-](+)$	$[-i](-)(+)$	$(-i)(-)[+]$	$[-i][-][+]$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$(-i)(+)[-]$	$[-i][+][-]$	$(-i)[+](-)$	$[-i](+)(-)$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$[+i][-][-]$	$(+i)(-)[-]$	$[+i](-)(-)$	$(+i)[-](-)$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$	$\rightarrow$	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ 03 12 56	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ 03 12 56	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ 03 12 56	S^{03}	S^{12}	S^{56}
even II ${}^{II}\mathcal{A}^m_f$	1	$[-i](+)(+)$	$(-i)[+](+)$	$[-i][+][+]$	$(-i)(+)[+]$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$(+i)[-](+)$	$[+i](-)(+)$	$(+i)(-)[+]$	$[+i][-][+]$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$(+i)(+)[-]$	$[+i][+][-]$	$(+i)[+](-)$	$[+i](+)(-)$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$[-i][-][-]$	$(-i)(-)[-]$	$[-i](-)(-)$	$(-i)[-](-)$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

$\hat{b}^{m=2\dagger}_{f=2}$, presented in *odd I* as the second member of the second column. (This can easily be evaluated by taking into account Eq. (47) for any m.)

$$ {}^{I}\hat{\mathcal{A}}^{1\dagger}_4 (\equiv \stackrel{03}{(+i)}\stackrel{12}{(+)}\stackrel{56}{[+]}) *_A \hat{b}^{2\dagger}_2 (\equiv \stackrel{03}{(-i)}\stackrel{12}{(-)}\stackrel{56}{(+)}) \rightarrow \hat{b}^{1\dagger}_2 (\equiv \stackrel{03}{[+i]}\stackrel{12}{[+]}\stackrel{56}{(+)}) \,. \tag{28} $$

The sign $\rightarrow$ means that the relation is valid up to the constant. The Hermitian conjugated partner of ${}^{I}\hat{\mathcal{A}}^{1\dagger}_4$ is ${}^{I}\hat{\mathcal{A}}^{2\dagger}_3$.

Let us check the Cartan subalgebra quantum numbers of this "scattering": ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{4}$ has $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56}) = (i, 1, 0)$, $\hat{b}^{2\dagger}_{2}$ has $(S^{03}, S^{12}, S^{56}) = (-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ and $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}) = (-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$, and $\hat{b}^{1\dagger}_{2}$ has $(S^{03}, S^{12}, S^{56}) = (\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}) = (-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$. This means that Clifford even "basis vector" changes the family members quantum numbers of the Clifford odd "basis vector", leaving the family quantum numbers unchanged.

One can find that the algebraic application, $*_A$, of ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{3} (\equiv \overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[+]})$ on $\hat{b}^{1\dagger}_{1}$ leads to the same family member of the same family $f = 1$, namely to $\hat{b}^{1\dagger}_{1}$.

Calculating the eigenvalues of the Cartan subalgebra members, Eq. (8), before and after the algebraic multiplication, $*_A$, assures us that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$ carry the integer eigenvalues of the Cartan subalgebra members, namely of $\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, since they transfer to the Clifford odd "basis vector" integer eigenvalues of the Cartan subalgebra members, changing the Clifford odd "basis vector" into another Clifford odd "basis vector" of the same family.

We, therefore, confirm that the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$, $m = 1, 2, 3, 4$, on $\hat{b}^{1\dagger}_{1}$ transforms $\hat{b}^{1\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}$, $m = (1, 2, 3, 4)$. Similarly we find that the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{4}$, $m = (1, 2, 3, 4)$ on $\hat{b}^{2\dagger}_{1}$ transforms $\hat{b}^{2\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}, m = (1, 2, 3, 4)$. The algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{2}$, $m = (1, 2, 3, 4)$ on $\hat{b}^{3\dagger}_{1}$ transforms $\hat{b}^{3\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}, m = (1, 2, 3, 4)$. And the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{1}$, $m = (1, 2, 3, 4)$ on $\hat{b}^{4\dagger}_{1}$ transforms $\hat{b}^{4\dagger}_{1}$ into $\hat{b}^{m\dagger}_{1}, m = (1, 2, 3, 4)$.

One easily checks Eq. (21) if taking into account Eq. (47); like: $\hat{b}^{1\dagger}_{1} *_A {}^{I}\hat{\mathcal{A}}^{m}_{4} = 0$, $(m = (1, 2, 3, 4))$, since either $((\overset{03}{+i}))^2 = 0$ or $\overset{12}{[+]} *_A \overset{12}{[-]} = 0$ or $\overset{56}{[+]}\overset{56}{(-)} = 0$.

Similarly, one can check Eq. (22) by evaluating, for example, ${}^{II}\hat{\mathcal{A}}^{m}_{4} *_A \hat{b}^{1\dagger}_{1}$, since either $\overset{12}{(+)} *_A \overset{12}{[+]} = 0$ or $\overset{12}{[-]}\overset{56}{[+]} = 0$.

Let us check the validity of Eq. (23) on the case: $\hat{b}^{4\dagger}_{1} *_A {}^{II}\hat{\mathcal{A}}^{m}_{4} = \hat{b}^{4\dagger}_{3}$ for $m = 1$, and zero for $m = (2, 3, 4)$, while $\hat{b}^{4\dagger}_{1} *_A {}^{II}\hat{\mathcal{A}}^{1}_{f} = (\hat{b}^{4\dagger}_{4}, \hat{b}^{4\dagger}_{2}, \hat{b}^{4\dagger}_{1}, \hat{b}^{4\dagger}_{3})$ for $f = (1, 2, 3, 4)$. All ${}^{II}\hat{\mathcal{A}}^{m}_{f}$ giving non zero contributions, keep the family member quantum numbers of the Clifford odd "basis vectors" unchanged, changing the family quantum number. All the rest give zero contribution.

The statements of Eq. (20), (21), (22), (23), are, therefore, demonstrated on the case of $d = (5 + 1)$.

The Cartan subalgebra has in $d = (5 + 1)$-dimensional space 3 members. To illustrate that the Clifford even "basis vectors" have the properties of the gauge fields of the corresponding Clifford odd "basis vectors" let us study properties of the $SU(3) \times U(1)$ subgroups of the Clifford odd and Clifford even "basis vectors". We need the relations between S^{ab} and $(\tau^3, \tau^8, \tau`)$

$$
\begin{aligned}
&\tau^3 := \frac{1}{2}(-S^{12} - iS^{03}), \qquad \tau^8 = \frac{1}{2\sqrt{3}}(-iS^{03} + S^{12} - 2S^{56}), \\
&\tau' = -\frac{1}{3}(-iS^{03} + S^{12} + S^{56}). \qquad (29)
\end{aligned}
$$

The corresponding relations for $(\tilde{\tau}^3, \tilde{\tau}^8, \tilde{\tau}')$ can be read from Eq. (29), if replacing S^{ab} by $\tilde{S}^{ab}$.

The corresponding relations for superposition of the Cartan subalgebra elements (τ', τ^3, τ^8) for $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ follow if in Eq. (29) S^{ab} is replaced by $\mathcal{S}^{ab}$.

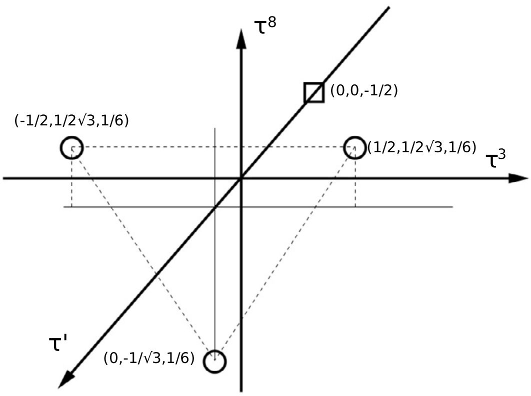

Fig. 1. The representations of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, the properties of which appear in Tables (1, 2) for the Clifford odd "basis vectors", are presented. (τ^3, τ^8, τ') can be calculated if using Eq. (29). On the abscissa axis, on the ordinate axis and on the third axis, the eigenvalues of the superposition of the three Cartan subalgebra members, (τ^3, τ^8, τ'), are presented. One notices one triplet, denoted by $\bigcirc$ with the values $\tau' = \frac{1}{6}$, $(\tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6})$, $(\tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}, \tau' = \frac{1}{6})$, $(\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{1}{6})$, respectively, and one singlet denoted by the square. $(\tau^3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2})$. The triplet and the singlet appear in four families, with the family quantum numbers presented in the last three columns of Table 2.

In Tables 2, 3 the Clifford odd and even "basis vectors" ($\hat{b}^{m\dagger}_f$ and ${}^I\hat{\mathcal{A}}^m_f$, respectively) are presented as products of nilpotents (odd number of nilpotents for $\hat{b}^{m\dagger}_f$ and even number of nilpotents for ${}^I\hat{\mathcal{A}}^m_f$) and projectors: Like in Table 1. Besides the eigenvalues of the Cartan subalgebra members of Eq. (8) also $(\tau^3, \tau^8, \tau^{‘})$ are added on both tables. In Table 2 also $(\tilde{\tau}^3, \tilde{\tau}^8, \tilde{\tau}^{‘})$ are written. In Fig. 1 only one family is presented; all four families have the same $(\tau^3, \tau^8, \tau^{‘})$, they only distinguish in $(\tilde{\tau}^3, \tilde{\tau}^8, \tilde{\tau}^{‘})$.

The corresponding table for the Clifford even "basis vectors" ${}^{II}\hat{\mathcal{A}}^m_f$ are not presented. ${}^{II}\hat{\mathcal{A}}^m_f$ carry, namely, the same quantum numbers $(\tau^3, \tau^8, \tau^{‘})$ as ${}^I\hat{\mathcal{A}}^m_f$. There are only products of nilpotents and projectors which distinguish among ${}^I\hat{\mathcal{A}}^m_f$ and ${}^{II}\hat{\mathcal{A}}^m_f$, causing differences in properties with respect to the Clifford odd "basis vectors"; ${}^{II}\hat{\mathcal{A}}^{m'}_{f^{‘}}$ transform $\hat{b}^{m\dagger}_f$ with a family member m of particular family f into $\hat{b}^{m\dagger}_{f''}$ of the same family member m of another family f''. ${}^I\hat{\mathcal{A}}^m_f$ transform a family member of particular family $\hat{b}^{m'\dagger}_{f^{‘}}$ into another family member m of the same family $\hat{b}^{m\dagger}_{f^{‘}}$. (Let us remind the reader that the $SO(5,1)$ group and the $SU(3), U(1)$ subgroups have the same number of commuting operators, but different number of generators; $SO(5,1)$ has 15 generators, $SU(3)$ and $U(1)$ have together 9 generators.)

In the case that the group $SO(5,1)$ — manifesting as $SU(3) \times U(1)$ and representing the colour group with quantum numbers (τ^3, τ^8) and the "fermion" group with the quantum number τ' — is embedded into $SO(13,1)$ the triplet would represent quarks (and antiquarks), and the singlet leptons (and antileptons).

The corresponding gauge fields, presented in Table 3 and Fig. 2, if belonging to the sextet, would transform the triplet of quarks among themselves, changing the colour and leaving the "fermion" quantum number equal to $\frac{1}{6}$.

Table 2
The "basis vectors" $\hat{b}^{m\dagger}_f$ are presented for $d = (5+1)$-dimensional case. Each $\hat{b}^{m\dagger}_f$ is a product of projectors and of an odd number of nilpotents and is the "eigenvector" of all the Cartan subalgebra members, (S^{03}, S^{12}, S^{56}) and ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$), Eq. (8), m counts the members of each family, while f determines the family quantum numbers (the eigenvalues of ($\tilde{S}^{03}$, $\tilde{S}^{12}$, $\tilde{S}^{56}$)). This table also presents in the columns ($8^{th}, 9^{th}, 10^{th}$) the eigenvalues of the three commuting operators (τ^3, τ^8 and τ') of the subgroups $SU(3) \times U(1)$, Eq. (29), as well as (in the last three columns) the corresponding ($\tilde{\tau}^3, \tilde{\tau}^8, \tilde{\tau}'$). $\Gamma^{(3+1)} = i\gamma^0\gamma^1\gamma^2\gamma^3$ is written in the 7^{th} column. $\Gamma^{(5+1)} = -1$ $(= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6)$. Operators $\hat{b}^{m\dagger}_f$ and $\hat{b}^{m}_f$ fulfil the anti-commutation relations of Eq. (16).

f	m	$\hat{b}^{m\dagger}_f$	S^{03}	S^{12}	S^{56}	Γ^{3+1}	τ^3	τ^8	τ'	$\tilde{S}^{03}$	$\tilde{S}^{12}$	$\tilde{S}^{56}$	$\tilde{\tau}^3$	$\tilde{\tau}^8$	$\tilde{\tau}'$
I	1	$\overset{03}{(+i)}\,\overset{12}{[+]}\,\vert\,\overset{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	2	$\overset{03}{[-i]}\,\overset{12}{(-)}\,\vert\,\overset{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	3	$\overset{03}{[-i]}\,\overset{12}{[+]}\,\vert\,\overset{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	4	$\overset{03}{(+i)}\,\overset{12}{(-)}\,\vert\,\overset{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
II	1	$\overset{03}{[+i]}\,\overset{12}{[+]}\,\vert\,\overset{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	2	$\overset{03}{(-i)}\,\overset{12}{(-)}\,\vert\,\overset{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	3	$\overset{03}{(-i)}\,\overset{12}{[+]}\,\vert\,\overset{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
	4	$\overset{03}{[+i]}\,\overset{12}{(-)}\,\vert\,\overset{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$
III	1	$\overset{03}{[+i]}\,\overset{12}{(+)}\,\vert\,\overset{56}{[+]}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	2	$\overset{03}{(-i)}\,\overset{12}{[-]}\,\vert\,\overset{56}{[+]}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	3	$\overset{03}{(-i)}\,\overset{12}{(+)}\,\vert\,\overset{56}{(-)}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
	4	$\overset{03}{[+i]}\overset{12}{[-]}\,\vert\,\overset{56}{(-)}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$
IV	1	$\overset{03}{(+i)}\,\overset{12}{(+)}\,\vert\,\overset{56}{(+)}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
	2	$\overset{03}{[-i]}\,\overset{12}{[-]}\,\vert\,\overset{56}{(+)}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
	3	$\overset{03}{[-i]}\,\overset{12}{(+)}\,\vert\,\overset{56}{[-]}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
	4	$\overset{03}{(+i)}\,\overset{12}{[-]}\,\vert\,\overset{56}{[-]}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$

Table 3 presents the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_f$ for $d = (5+1)$ with the properties:

i. They are products of an even number of nilpotents, $\overset{ab}{(k)}$, with the rest up to $\frac{d}{2}$ of projectors, $\overset{ab}{[k]}$.

ii. Nilpotents and projectors are eigenvectors of the Cartan subalgebra members $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$, Eq. (8), carrying the integer eigenvalues of the Cartan subalgebra members.

iii. They have their Hermitian conjugated partners within the same group of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_f$ (with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members).

iv. They have properties of the boson gauge fields. When the Clifford even "basis vectors", ${}^{I}\hat{\mathcal{A}}^{m\dagger}_f$, apply on the Clifford odd "basis vectors" (offering the description of the fermion fields)

Table 3
The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, each of them is the product of projectors and an even number of nilpotents, and each is the eigenvector of all the Cartan subalgebra members, $\mathcal{S}^{03}$, $\mathcal{S}^{12}$, $\mathcal{S}^{56}$, Eq. (8), are presented for $d = (5+1)$-dimensional case. Indexes m and f determine $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ different members ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$. In the third column the "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ which are Hermitian conjugated partners to each other (and can therefore annihilate each other) are pointed out with the same symbol. For example, with $\star\star$ are equipped the first member with $m = 1$ and $f = 1$ and the last member of $f = 3$ with $m = 4$. The sign $\bigcirc$ denotes the Clifford even "basis vectors" which are self-adjoint $({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f})^{\dagger} = {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}$. It is obvious that † has no meaning, since ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are self adjoint or are Hermitian conjugated partner to another ${}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}$. This table also represents the eigenvalues of the three commuting operators τ^3, τ^8 and τ' of the subgroups $SU(3) \times U(1)$.

f	m	$*$	${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$	$\mathcal{S}^{03}$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$	τ^3	τ^8	τ'
I	1	$\star\star$	$\overset{03}{[+i]}\overset{12}{(+)}\overset{56}{(+)}$	0	1	1	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\triangle$	$\overset{03}{(-i)}\overset{12}{[-]}\overset{56}{(+)}$	$-i$	0	1	$-\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\ddagger$	$\overset{03}{(-i)}\overset{12}{(+)}\overset{56}{[-]}$	$-i$	1	0	-1	0	0
	4	$\bigcirc$	$\overset{03}{[+i]}\overset{12}{[-]}\overset{56}{[-]}$	0	0	0	0	0	0
II	1	$\bullet$	$\overset{03}{(+i)}\overset{12}{[+]}\overset{56}{(+)}$	i	0	1	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{2}{3}$
	2	$\otimes$	$\overset{03}{[-i]}\overset{12}{(-)}\overset{56}{(+)}$	0	-1	1	$\frac{1}{2}$	$-\frac{3}{2\sqrt{3}}$	0
	3	$\bigcirc$	$\overset{03}{[-i]}\overset{12}{[+]}\overset{56}{[-]}$	0	0	0	0	0	0
	4	$\ddagger$	$\overset{03}{(+i)}\overset{12}{(-)}\overset{56}{[-]}$	i	-1	0	1	0	0
III	1	$\bigcirc$	$\overset{03}{[+i]}\overset{12}{[+]}\overset{56}{[+]}$	0	0	0	0	0	0
	2	$\odot\odot$	$\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{[+]}$	$-i$	-1	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{2}{3}$
	3	$\bullet$	$\overset{03}{(-i)}\overset{12}{[+]}\overset{56}{(-)}$	$-i$	0	-1	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
	4	$\star\star$	$\overset{03}{[+i]}\overset{12}{(-)}\overset{56}{(-)}$	0	-1	-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{2}{3}$
IV	1	$\odot\odot$	$\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[+]}$	i	1	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{2}{3}$
	2	$\bigcirc$	$\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[+]}$	0	0	0	0	0	0
	3	$\otimes$	$\overset{03}{[-i]}\overset{12}{(+)}\overset{56}{(-)}$	0	1	-1	$-\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0
	4	$\triangle$	$\overset{03}{(+i)}\overset{12}{[-]}\overset{56}{(-)}$	i	0	-1	$\frac{1}{2}$	$\frac{3}{2\sqrt{3}}$	0

they transform the Clifford odd "basis vectors" into another Clifford odd "basis vectors" of the same family, transferring to the Clifford odd "basis vectors" the integer spins with respect to the $SO(d-1, 1)$ group, while with respect to subgroups of the $SO(d-1, 1)$ group they transfer appropriate superposition of the eigenvalues (manifesting the properties of the adjoint representations of the corresponding subgroups.)

If, for example, ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{3}$ applies on a singlet $\hat{b}^{1\dagger}_{1}$ keeps the internal space of $\hat{b}^{1\dagger}_{1}$ unchanged (it can change only momentum), while if ${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}$ applies on $\hat{b}^{1\dagger}_{1}$ transforms it to a member of a triplet, to $\hat{b}^{2\dagger}_{1}$.

We can see that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{3}$ with $(m = 2, 3, 4)$, if applied on the $SU(3)$ singlet $\hat{b}^{1\dagger}_{4}$ with $(\tau' = -\frac{1}{2}, \tau^3 = 0, \tau^8 = 0)$, transforms it to $\hat{b}^{m(=2,3,4)\dagger}_{4}$, respectively, which are members of the $SU(3)$ triplet. All these Clifford even "basis vectors" have τ' equal to $\frac{2}{3}$, changing correspondingly $\tau' = -\frac{1}{2}$ into $\tau' = \frac{1}{6}$ and bringing the needed values of τ^3 and τ^8.

In Table 3 we find $(6 + 4)$ Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ with $\tau' = 0$. Six of them are Hermitian conjugated to each other — the Hermitian conjugated partners are denoted by the same geometric figure on the third column. Four of them are self-adjoint and correspondingly with $(\tau' = 0, \tau^3 = 0, \tau^8 = 0)$, denoted in the third column of Table 3 by $\bigcirc$. The rest 6 Clifford even "basis vectors" belong to one triplet with $\tau' = \frac{2}{3}$ and (τ^3, τ^8) equal to $[(0, -\frac{1}{\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (\frac{1}{2}, \frac{1}{2\sqrt{3}})]$ and one antitriplet with $\tau' = -\frac{2}{3}$ and $((\tau^3, \tau^8)$ equal to $[(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}), (\frac{1}{2}, -\frac{1}{2\sqrt{3}}), (0, \frac{1}{\sqrt{3}})]$.

Each triplet has Hermitian conjugated partners in anti-triplet and opposite. In Table 3 the Hermitian conjugated partners of the triplet and antitriplet are denoted by the same signum: $({}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}, {}^{I}\hat{\mathcal{A}}^{4\dagger}_{3})$ by $\star\star$, $({}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}, {}^{I}\hat{\mathcal{A}}^{3\dagger}_{3})$ by $\bullet$, and $({}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}, {}^{I}\hat{\mathcal{A}}^{1\dagger}_{4})$ by $\odot\odot$.

The octet, two triplets and four singlets are presented in Fig. 2.

Fig. 2 represents the $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members ${}^{I}\hat{\mathcal{A}}^{m}_{f}$ of the Clifford even "basis vectors" for the case that $d = (5 + 1)$. The properties of ${}^{I}\hat{\mathcal{A}}^{m}_{f}$ are presented also in Table 3. Manifesting the structure of subgroups $SU(3) \times U(1)$ of the group $SO(5, 1)$ they are represented as eigenvectors of the superposition of the Cartan subalgebra members $(\mathcal{S}^{03}, \mathcal{S}^{12}, \mathcal{S}^{56})$, that is with $\tau^3 = \frac{1}{2}(-\mathcal{S}^{12} - i\mathcal{S}^{03})$, $\tau^8 = \frac{1}{2\sqrt{3}}(\mathcal{S}^{12} - i\mathcal{S}^{03} - 2\mathcal{S}^{56})$, and $\tau' = -\frac{1}{3}(\mathcal{S}^{12} - i\mathcal{S}^{03} + \mathcal{S}^{56})$. There are four self adjoint Clifford even "basis vectors" with $(\tau^3 = 0, \tau^8 = 0, \tau' = 0)$, one sextet of three pairs Hermitian conjugated to each other, one triplet and one antitriplet with the members of the triplet Hermitian conjugated to the corresponding members of the antitriplet and opposite. These 16 members of the Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m}_{f}$ are the gauge fields "partners" of the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, presented in Fig. 1 for one of four families, anyone. The reader can check that the algebraic application of ${}^{I}\hat{\mathcal{A}}^{m}_{f}$, belonging to the triplet transforms applying on the Clifford odd singlet, denoted in Fig. 1 by a square, this singlet to one of the members of the triplet, denoted in Fig. 1 by the circle $\bigcirc$.

Looking at the boson fields ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ from the point of view of subgroups $SU(3) \times U(1)$ of the group $SO(5 + 1)$ we recognize in the part of fields forming the octet the colour gauge fields of quarks and leptons and antiquarks and antileptons. The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ transform when applying on the Clifford odd "basis vectors" $\hat{b}^{m'\dagger}_{f'}$ to another (or the same) member, keeping the family member unchanged.

We can check that although ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ have different structure of an even number of nilpotents, and the rest of the projectors than ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, having correspondingly different properties with respect to the Clifford odd "basis vectors": ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ transform $\hat{b}^{m'\dagger}_{f^{\cdot}}$ among the family members, keeping the family quantum numbers unchanged, ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ transform $\hat{b}^{m\dagger}_{f}$ into the same member of another family, keeping the family member's quantum number unchanged, both, ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$ do have the equivalent figure and equivalent $\mathbf{S}^{ab}$ and correspondingly also (τ^3, τ^8, τ') content.

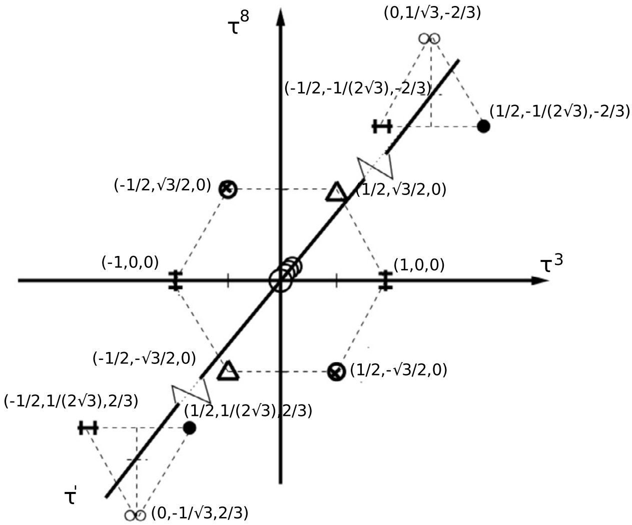

Fig. 2. The Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ in the case that $d=(5+1)$ are presented concerning the eigenvalues of the commuting operators of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5,1)$, Eq. (29): $(\tau^3,\ \tau^8,\ \tau')$. Their properties appear in Table 3. The abscissa axis carries the eigenvalues of τ^3, the ordinate axis carries the eigenvalues of τ^8 and the third axis carries the eigenvalues of τ'. One notices four singlets with $(\tau^3=0, \tau^8=0, \tau'=0)$, denoted by $\bigcirc$, representing four self adjoint Clifford even "basis vectors" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, with $(f=1, m=4)$, $(f=2, m=3)$, $(f=3, m=1)$, $(f=4, m=2)$, one sextet of three pairs, Hermitian conjugated to each other, with $\tau'=0$, denoted by $\triangle$ (${}^{I}\hat{\mathcal{A}}^{2\dagger}_{1}$ with $(\tau'=0, \tau^3=-\frac{1}{2}, \tau^8=-\frac{3}{2\sqrt{3}})$ and ${}^{I}\hat{\mathcal{A}}^{4\dagger}_{4}$ with $(\tau'=0, \tau^3=\frac{1}{2}, \tau^8=\frac{3}{2\sqrt{3}})$), by $\ddagger$ (${}^{I}\hat{\mathcal{A}}^{3\dagger}_{1}$ with $(\tau'=0, \tau^3=-1, \tau^8=0)$ and ${}^{I}\hat{\mathcal{A}}^{4\dagger}_{2}$ with $\tau'=0, \tau^3=1, \tau^8=0$)), and by $\otimes$ (${}^{I}\hat{\mathcal{A}}^{2\dagger}_{2}$ with $(\tau'=0, \tau^3=\frac{1}{2}, \tau^8=-\frac{3}{2\sqrt{3}})$ and ${}^{I}\hat{\mathcal{A}}^{3\dagger}_{4}$ with $(\tau'=0, \tau^3=-\frac{1}{2}, \tau^8=\frac{3}{2\sqrt{3}})$), and one triplet, denoted by $\star\star$ (${}^{I}\hat{\mathcal{A}}^{4\dagger}_{3}$ with $(\tau'=\frac{2}{3}, \tau^3=\frac{1}{2}, \tau^8=\frac{1}{2\sqrt{3}})$), by $\bullet$ (${}^{I}\hat{\mathcal{A}}^{3\dagger}_{3}$ with $(\tau'=\frac{2}{3}, \tau^3=-\frac{1}{2}, \tau^8=\frac{1}{2\sqrt{3}})$), and by $\odot\odot$ (${}^{I}\hat{\mathcal{A}}^{2\dagger}_{3}$ with $(\tau'=\frac{2}{3}, \tau^3=0, \tau^8=-\frac{1}{\sqrt{3}})$), as well as one antitriplet, Hermitian conjugated to triplet, denoted by $\star\star$ (${}^{I}\hat{\mathcal{A}}^{1\dagger}_{1}$ with $(\tau'=-\frac{2}{3}, \tau^3=-\frac{1}{2}, \tau^8=-\frac{1}{2\sqrt{3}})$), by $\bullet$ (${}^{I}\hat{\mathcal{A}}^{1\dagger}_{2}$ with $(\tau'=-\frac{2}{3}, \tau^3=\frac{1}{2}, \tau^8=-\frac{1}{2\sqrt{3}})$), and by $\odot\odot$ (${}^{I}\hat{\mathcal{A}}^{4\dagger}_{1}$ with $(\tau'=-\frac{2}{3}, \tau^3=0, \tau^8=\frac{1}{\sqrt{3}})$).

Let us anyhow demonstrate properties of "scattering" of $\hat{b}^{m\dagger}_{f}$ on ${}^{II}\hat{\mathcal{A}}^{m'\dagger}_{f'}$, paying attention on $SU(3)$ and $U(1)$ substructure of $SO(5,1)$.

Let us look at the "scattering" of the kind of Eq. (28)

$$\hat{b}^{2\dagger}_{2}(\equiv\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(+)}) *_A {}^{II}\hat{\mathcal{A}}^{3\dagger}_{1}(\equiv\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[-]}) \rightarrow \hat{b}^{2\dagger}_{4}(\equiv\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)})\,, \tag{30}$$

$\hat{b}^{2\dagger}_{2}(\equiv\overset{03}{(-i)}\overset{12}{(-)}\overset{56}{(+)})$ has $(\tau^3=0, \tau^8=-\frac{1}{\sqrt{3}}, \tau'=\frac{1}{6})$ and $(\tilde{\tau}^3=0, \tilde{\tau}^8=-\frac{1}{\sqrt{3}}, \tilde{\tau}'=\frac{1}{6})$. $\hat{b}^{2\dagger}_{4}(\equiv\overset{03}{[-i]}\overset{12}{[-]}\overset{56}{(+)})$ has $(\tau^3=0, \tau^8=-\frac{1}{\sqrt{3}}, \tau'=\frac{1}{6})$ and $(\tilde{\tau}^3=0, \tilde{\tau}^8=0, \tilde{\tau}'=-\frac{1}{2})$. ${}^{II}\hat{\mathcal{A}}^{1\dagger}_{4}(\equiv\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{[-]})$ has $(\tau^3=0, \tau^8=\frac{1}{\sqrt{3}}, \tau'=-\frac{2}{3})$.

If $\hat{b}^{2\dagger}_{2}$ absorbs ${}^{II}\hat{\mathcal{A}}^{3\dagger}_{4}(\equiv\stackrel{03}{[+i]}\stackrel{12}{(+)}\stackrel{56}{(-)})$ with $(\tau^3=-\frac{1}{2},\tau^8=\frac{3}{2\sqrt{3}},\tau'=0)$ becomes $\hat{b}^{2\dagger}_{3}(\equiv\stackrel{03}{(-i)}\stackrel{12}{[-]}\stackrel{56}{[+]})$ with quantum numbers $(\tau^3=0,\tau^8=-\frac{1}{\sqrt{3}},\tau'=\frac{1}{6})$ and $(\tilde{\tau}^3=-\frac{1}{2},\tilde{\tau}^8=\frac{1}{2\sqrt{3}},\tilde{\tau}'=\frac{1}{6})$.

${}^{II}\hat{\mathcal{A}}^{3\dagger}_{4}$ transfers its quantum numbers to $\hat{b}^{2\dagger}_{2}$, changing family and leaving the family member m unchanged.

2.4. Second quantized fermion and boson fields with internal spaces described by Clifford "basis vectors" in even dimensional spaces

We learned in the previous Subsects. (2.2, 2.3) that in even dimensional spaces ($d=2(2n+1)$ or $d=4n$) the Clifford odd and the Clifford even "basis vectors", which are the superposition of the Clifford odd and the Clifford even products of γ^a's, respectively, offer the description of the internal spaces of fermion and boson fields.

The Clifford odd algebra offers $2^{\frac{d}{2}-1}$ "basis vectors" $\hat{b}^{m\dagger}_{f}$, appearing in $2^{\frac{d}{2}-1}$ families (with the family quantum numbers determined by $\tilde{S}^{ab}=\frac{i}{2}\{\tilde{\gamma}^a,\tilde{\gamma}^b\}_-$), which, together with their $2^{\frac{d}{2}-1}\times\, 2^{\frac{d}{2}-1}$ Hermitian conjugated partners $\hat{b}^{m}_{f}$ fulfil the postulates for the second quantized fermion fields, Eq. (16) in this paper, Eq. (26) in Ref. [14], explaining the second quantization postulate of Dirac.

The Clifford even algebra offers $2^{\frac{d}{2}-1}\times\, 2^{\frac{d}{2}-1}$ "basis vectors" of ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$, and the same number of ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}$, with the properties of the second quantized boson fields manifesting as the gauge fields of fermion fields described by the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$. The commutation relations of ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i=(I,II)$, are commented in the last paragraph of App. A on a simple case of $d=(3+1)$. The subgroup structure of $SU(3)$ can be recognized on Fig. 2, leading to the commutation relations of the observed colour boson gauge fields. However, further studies are needed to recognize what new this way of describing internal spaces of fermion and boson fields with the Clifford algebra is offering.

The Clifford odd and the Clifford even "basis vectors" are chosen to be products of nilpotents, $\stackrel{ab}{(k)}$ (with the odd number of nilpotents if describing fermions and the even number of nilpotents if describing bosons), and projectors, $\stackrel{ab}{[k]}$. Nilpotents and projectors are (chosen to be) eigenvectors of the Cartan subalgebra members of the Lorentz algebra in the internal space of S^{ab} for the Clifford odd "basis vectors" and of $\mathcal{S}^{ab}(=S^{ab}+\tilde{S}^{ab})$ for the Clifford even "basis vectors".

To define the creation operators, for fermions or bosons, besides the "basis vectors" defining the internal space of fermions and bosons, the basis in ordinary space in momentum or coordinate representation is needed. Here Ref. [14], Subsect. 3.3 and App. J is overviewed.

Let us introduce the momentum part of the single-particle states. (The extended version is presented in Ref. [14] in Subsect. 3.3 and App. J.)

$$
\begin{aligned}
|\vec{p}> &= \hat{b}^{\dagger}_{\vec{p}}\,|\,0_p>\,,\quad <\vec{p}\,| = <0_p\,|\,\hat{b}_{\vec{p}}\,,\\
<\vec{p}\,|\,\vec{p}'> &= \delta(\vec{p}-\vec{p}') = <0_p\,|\hat{b}_{\vec{p}}\;\hat{b}^{\dagger}_{\vec{p}'}|\,0_p>\,,\\
&\quad \text{pointing out}
\end{aligned}
$$

$$< 0_p \,|\hat{b}_{\vec{p'}}\, \hat{b}^{\dagger}_{\vec{p}} \,|\, 0_p \,> = \delta(\vec{p'} - \vec{p})\,, \tag{31}$$

with the normalization $< 0_p \,|\, 0_p \,>= 1$. While the quantized operators $\hat{\vec{p}}$ and $\hat{\vec{x}}$ commute $\{\hat{p}^i \,, \hat{p}^j\}_- = 0$ and $\{\hat{x}^k \,, \hat{x}^l\}_- = 0$, it follows for $\{\hat{p}^i \,, \hat{x}^j\}_- = i\eta^{ij}$. One correspondingly finds

$$\begin{aligned}
&< \vec{p}\,|\,\vec{x}\,> = < 0_{\vec{p}}\,|\,\hat{b}_{\vec{p}}\,\hat{b}^{\dagger}_{\vec{x}}|0_{\vec{x}}\,> = (< 0_{\vec{x}}\,|\,\hat{b}_{\vec{x}}\,\hat{b}^{\dagger}_{\vec{p}}\,|0_{\vec{p}}\,>)^{\dagger}\\
&< 0_{\vec{p}}\,|\{\hat{b}^{\dagger}_{\vec{p}}\,, \hat{b}^{\dagger}_{\vec{p}\,'}\}_-|0_{\vec{p}}\,> = 0\,, \quad < 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}\,'}\}_-|0_{\vec{p}}\,> = 0\,, \quad < 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}}, \hat{b}^{\dagger}_{\vec{p}\,'}\}_-|0_{\vec{p}}\,> = 0,\\
&< 0_{\vec{x}}\,|\{\hat{b}^{\dagger}_{\vec{x}}, \hat{b}^{\dagger}_{\vec{x}\,'}\}_-|0_{\vec{x}}\,> = 0\,, \quad < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}\,'}\}_-|0_{\vec{x}}\,> = 0\,, \quad < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}}, \hat{b}^{\dagger}_{\vec{x}\,'}\}_-|0_{\vec{x}}\,> = 0\,,\\
&< 0_{\vec{p}}\,|\{\hat{b}_{\vec{p}}, \hat{b}^{\dagger}_{\vec{x}}\}_-|0_{\vec{x}}\,> = e^{i\,\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,, \quad < 0_{\vec{x}}\,|\{\hat{b}_{\vec{x}}, \hat{b}^{\dagger}_{\vec{p}}\}_-|0_{\vec{p}}\,> = e^{-i\,\vec{p}\cdot\vec{x}}\frac{1}{\sqrt{(2\pi)^{d-1}}}\,.
\end{aligned} \tag{32}$$

The internal space of either fermion or boson fields has the finite number of "basis vectors", $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ for fermions (and the same number of their Hermitian conjugated partners), and twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ for bosons, the momentum basis is continuously infinite.

The creation operators for either fermions or bosons must be tensor products, $*_T$, of both contributions, the "basis vectors" describing the internal space of fermions or bosons and the basis in ordinary momentum or coordinate space.

The creation operators for a free massless fermion of the energy $p^0 = |\vec{p}|$, belonging to a family f and to a superposition of family members m applying on the vacuum state $|\psi_{oc}> *_T |0_{\vec{p}}>$ can be written as ([14], Subsect. 3.3.2, and the references therein)

$$\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p}) = \sum_{m} c^{sm}{}_{f}(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}} *_T \hat{b}^{m\dagger}_{f}\,, \tag{33}$$

where the vacuum state for fermions $|\psi_{oc}> *_T |0_{\vec{p}}>$ includes both spaces, the internal part, Eq. (15), and the momentum part, Eq. (31) (in a tensor product for a starting single particle state with zero momentum, from which one obtains the other single fermion states of the same "basis vector" by the operator $\hat{b}^{\dagger}_{\vec{p}}$ which pushes the momentum by an amount $\vec{p}$[10]).

The creation operators and annihilation operators for fermion fields fulfil the anti-commutation relations for the second quantized fermion fields[11].[12]

[10] The creation operators and their Hermitian conjugated annihilation operators in the coordinate representation can be read in [14] and the references therein: $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{x}, x^0) = \sum_m \hat{b}^{m\dagger}_{f} *_T \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, c^{sm}{}_{f}\,(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}}\, e^{-i(p^0x^0 - \varepsilon\vec{p}\cdot\vec{x})}$ ([14], subsect. 3.3.2., Eqs. (55,57,64) and the references therein).

[11] Let us evaluate: $< 0_{\vec{p}}\,|\{\hat{\mathbf{b}}^{s'}_{f`}(\vec{p'})\,, \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_+ |\psi_{oc}> |0_{\vec{p}}> = \delta^{ss'}\delta_{ff'}\,\delta(\vec{p}' - \vec{p}) \cdot |\psi_{oc}> = < 0_{\vec{p}}\,|\hat{\mathbf{b}}^{s'}_{f`}\,\hat{\mathbf{b}}^{s\dagger}_{f}\,\hat{b}_{\vec{p}'}\hat{b}^{\dagger}_{\vec{p}} + \hat{b}^{\dagger}_{\vec{p}}\hat{b}_{\vec{p}'}\,\hat{\mathbf{b}}^{s\dagger}_{f}\,\hat{\mathbf{b}}^{s'}_{f`}\,|\psi_{oc}> |0_{\vec{p}}> = < 0_{\vec{p}}\,|\hat{\mathbf{b}}^{s'}_{f`}\,\hat{\mathbf{b}}^{s\dagger}_{f}\,\hat{b}_{\vec{p}'}\hat{b}^{\dagger}_{\vec{p}}\,|\psi_{oc}> |0_{\vec{p}}>$, since, according to Eq. (16), $\hat{\mathbf{b}}^{s'}_{f`}\,|\psi_{oc}> = 0$.

Let us demonstrate for free fields $|\vec{p}> = e^{-i\,\vec{p}\cdot\vec{x}}|0_p> = \hat{b}^{\dagger}_{\vec{p}}\,|0_p>$, $<\vec{p}| = < 0_p|e^{i\,\vec{p}\cdot\vec{x}} = < 0_p|\,\hat{b}_{\vec{p}}$
$< \vec{p'}|\vec{p}> = < 0_p|\,\hat{b}_{\vec{p'}}\,\hat{b}^{\dagger}_{\vec{p}}\,|0_p> = \delta(\vec{p'} - \vec{p})$, $< -\vec{p}'|-\vec{p}> = < 0_p|\,\hat{b}^{\dagger}_{\vec{p'}}\,\hat{b}_{\vec{p}}\,|0_p> = \delta(-\vec{p}' - (-\vec{p})) = \delta(\vec{p} - \vec{p}')$, consequently $< 0_p|\{\,\hat{b}_{\vec{p}}\,, \hat{b}^{\dagger}_{\vec{p'}}\}_-\,|0_p> = 0$.

[12] Two fermion states (formed from two creation operators applying on the vacuum state) with the orthogonal basis part in ordinary space (with two different momenta in ordinary space in the case of free fields) "do not meet"; correspondingly, each can carry the same "basis vector". They must differ in the internal basis if they have the identical ordinary part of the basis. (Otherwise, the tensor product, $*_{T_H}$, of such two fermion states is zero.) Illustration: Let us treat an atom with many electrons. Each electron has a spin of either $1/2$ or $-1/2$. Their orthogonal basis in ordinary space allows them

$$
\begin{aligned}
< 0_{\vec{p}}\, |\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{p}'), \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+} |\psi_{oc} > |0_{\vec{p}} > &= \delta^{ss'} \delta_{ff'}\, \delta(\vec{p}' - \vec{p}) \cdot |\psi_{oc} >, \\
\{\hat{\mathbf{b}}^{s'}_{f'}(\vec{p}'), \hat{\mathbf{b}}^{s}_{f}(\vec{p})\}_{+} |\psi_{oc} > |0_{\vec{p}} > &= 0 \cdot |\psi_{oc} > |0_{\vec{p}} >, \\
\{\hat{\mathbf{b}}^{s'\dagger}_{f'}(\vec{p}'), \hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\}_{+} |\psi_{oc} > |0_{\vec{p}} > &= 0 \cdot |\psi_{oc} > |0_{\vec{p}} >, \\
\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})\, |\psi_{oc} > |0_{\vec{p}} > &= |\psi^{s}_{f}(\vec{p}) >, \\
\hat{\mathbf{b}}^{s}_{f}(\vec{p})\, |\psi_{oc} > |0_{\vec{p}} > &= 0 \cdot\, |\psi_{oc} > |0_{\vec{p}} >, \\
|p^0| &= |\vec{p}|\,.
\end{aligned} \tag{34}
$$

The creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{\mathbf{b}}^{s}_{f}(\vec{p})$, creating and annihilating the single fermion states, respectively, fulfil when applying the vacuum state, $|\psi_{oc} > *_T |0_{\vec{p}} >$, the anti-commutation relations for the second quantized fermions, postulated by Dirac (Ref. [14], Subsect. 3.3.1, Sect. 5).[13]

To write the creation operators for boson fields, we must take into account that boson gauge fields have the space index α, describing the α component of the boson field in the ordinary space.[14] We, therefore, add the space index α as follows.

$$
{}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}\alpha}(\vec{p}) = \hat{b}^{\dagger}_{\vec{p}} *_T \,{}^{i}\mathcal{C}^{m}{}_{f\alpha}\, {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}\,, i = (I, II)\,. \tag{35}
$$

We treat free massless bosons of momentum $\vec{p}$ and energy $p^0 = |\vec{p}|$ and of particular “basis vectors” ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$’s which are eigenvectors of all the Cartan subalgebra members,[15] ${}^{i}\mathcal{C}^{m}{}_{f\alpha}$ carry the space index α of the boson field. Creation operators operate on the vacuum state $|\psi_{oc_{ev}} > *_T |0_{\vec{p}} >$ with the internal space part just a constant, $|\psi_{oc_{ev}} >= |\,1 >$, and for a starting single boson state with zero momentum from which one obtains the other single boson states with the same “basis vector” by the operators $\hat{b}^{\dagger}_{\vec{p}}$ which push the momentum by an amount $\vec{p}$, making also ${}^{i}\mathcal{C}^{m}{}_{f\alpha}$ depending on $\vec{p}$.

For the creation operators for boson fields in a coordinate representation one finds using Eqs. (31), (32)

$$
{}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}\alpha}(\vec{x}, x^0) = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}}\, {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}(\vec{p})\, e^{-i(p^0x^0 - \varepsilon \vec{p}\cdot\vec{x})}|_{p^0=|\vec{p}|}\,, i = (I, II)\,. \tag{36}
$$

to have the internal spin $\pm 1/2$ (leading to total angular momentum either $\pm 1/2$ or larger due to the angular momentum in ordinary space). As mentioned in the introduction section in **a.iii.** the Hilbert space of the second quantized fermion states is represented by the tensor products, $*_{T_H}$, of all possible members of creation operators from zero to infinity applying on the simple vacuum state. For any of these members the scalar product is obtained by multiplying from the left hand side by their Hermitian conjugated partner.

[13] The anti-commutation relations of Eq. (34) are valid also if we replace the vacuum state, $|\psi_{oc} > |0_{\vec{p}} >$, by the Hilbert space of the Clifford fermions generated by the tensor products multiplication, $*_{T_H}$, of any number of the Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is, of any number of $\hat{\mathbf{b}}^{s\,\dagger}_{f}(\vec{p})$ of any $(s, f, \vec{p})$), Ref. ([14], Sect. 5.).

[14] In the *spin-charge-family* theory the Higgs’s scalars origin in the boson gauge fields with the vector index $(7, 8)$, Ref. ([14], Sect. 7.4.1, and the references therein).

[15] In the general case, the energy eigenstates of bosons are in a superposition of ${}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}}$, for either $i = I$ or $i = II$. One example, which uses the superposition of the Cartan subalgebra eigenstates manifesting the $SU(3) \times U(1)$ subgroups of the group $SO(5, 1)$, is presented in Fig. 2.

To understand what new the Clifford algebra description of the internal space of fermion and boson fields, Eqs. (35), (36), (33), bring to our understanding of the second quantized fermion and boson fields and what new can we learn from this offer, we need to relate $\sum_{ab} c^{ab}\omega_{ab\alpha}$ and $\sum_{mf} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{I}\mathcal{C}^{m}{}_{f\alpha}$, recognizing that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{I}\mathcal{C}^{m}{}_{f\alpha}$ are eigenstates of the Cartan subalgebra members, while $\omega_{ab\alpha}$ are not. And, equivalently, we need to relate $\sum_{ab} \tilde{c}^{ab}\tilde{\omega}_{ab\alpha}$ and $\sum_{mf} {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{II}\mathcal{C}^{m}{}_{f\alpha}$.

The gravity fields, the vielbeins and the two kinds of spin connection fields, $f^{a}{}_{\alpha}$, $\omega_{ab\alpha}$, $\tilde{\omega}_{ab\alpha}$, respectively, are in the *spin-charge-family* theory (unifying spins, charges and families of fermions and offering not only the explanation for all the assumptions of the *standard model* but also for the increasing number of phenomena observed so far) the only boson fields in $d=(13+1)$, observed in $d=(3+1)$ besides as gravity also as all the other boson fields with the Higgs's scalars included [11].

We, therefore, need to relate:

$$\begin{aligned}
\{\frac{1}{2}\sum_{ab} S^{ab}\,\omega_{ab\alpha}\}\sum_{m}\beta^{mf}\,\hat{\mathbf{b}}^{m\dagger}_{f}(\vec{p})\ \text{related to}\ &\{\sum_{m'f'} {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,\mathcal{C}^{m'f'}_{\alpha}\}\sum_{m}\beta^{mf}\,\hat{\mathbf{b}}^{m\dagger}_{f}(\vec{p})\,,\\
&\forall f\ \text{and}\ \forall\,\beta^{mf}\,,\\
\mathcal{S}^{cd}\sum_{ab}(c^{ab}{}_{mf}\,\omega_{ab\alpha})\ \text{related to}\ &\mathcal{S}^{cd}\,({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,\mathcal{C}^{mf}_{\alpha})\,,\\
&\forall\,(m,f),\\
&\forall\ \text{Cartan subalgebra member}\,\mathcal{S}^{cd}\,.
\end{aligned}\tag{37}$$

Let be repeated that ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ are chosen to be the eigenvectors of the Cartan subalgebra members, Eq. (8). Correspondingly we can relate a particular ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{I}\mathcal{C}^{m}{}_{f\alpha}$ with such a superposition of $\omega_{ab\alpha}$'s, which is the eigenvector with the same values of the Cartan subalgebra members as there is a particular ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\mathcal{C}^{mf}_{\alpha}$. We can do this in two ways:

i. Using the first relation in Eq. (37). On the left hand side of this relation S^{ab}'s apply on $\hat{b}^{m\dagger}_{f}$ part of $\hat{\mathbf{b}}^{m\dagger}_{f}(\vec{p})$. On the right hand side ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ apply as well on the same "basis vector" $\hat{b}^{m\dagger}_{f}$.

ii. Using the second relation, in which $\mathcal{S}^{cd}$ apply on the left hand side on $\omega_{ab\alpha}$'s,

$$\mathcal{S}^{cd}\sum_{ab} c^{ab}{}_{mf}\,\omega_{ab\alpha}=\sum_{ab} c^{ab}{}_{mf}\, i\,(\omega_{cb\alpha}\eta^{ad}-\omega_{db\alpha}\eta^{ac}+\omega_{ac\alpha}\eta^{bd}-\omega_{ad\alpha}\eta^{bc}),\tag{38}$$

on each $\omega_{ab\alpha}$ separately; $c^{ab}{}_{mf}$ are constants to be determined from the second relation, where on the right-hand side of this relation $\mathcal{S}^{cd}(=S^{cd}+\tilde{S}^{cd})$ apply on the "basis vector" ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}$ of the corresponding gauge field.[16]

We must treat equivalently also ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\, {}^{II}\mathcal{C}^{m}{}_{f\alpha}$ and $\tilde{\omega}_{ab\alpha}$.

Let us conclude this section by pointing out that either the Clifford odd "basis vectors", $\hat{b}^{m\dagger}_{f}$, or the Clifford even "basis vectors", ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}, i=(I,II)$, have each in any even d, $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members, while $\omega_{ab\alpha}$ as well as $\tilde{\omega}_{ab\alpha}$ have each for a particular α $\frac{d}{2}(d-1)$ members. It is needed

[16] The reader can find the relation of Eq. (37) demonstrated for the case $d=3+1$ in Ref. [15] at the end of Sect. 3.

to find out what new this difference brings into the unifying theories of the Kaluza-Klein-like kind to what the *spin-charge-family* belongs.

3. Conclusions

In the *spin-charge-family* theory [6,8,11,9,22,12,14] the Clifford odd algebra describes the internal space of fermion fields. The Clifford odd "basis vectors" — the superposition of odd products of γ^a's — in a tensor product with the basis in ordinary space form the creation and annihilation operators, in which the anti-commutativity of the "basis vectors" is transferred to the creation and annihilation operators for fermions, explaining the second quantization postulates for fermion fields.

The Clifford odd "basis vectors" have all the properties of fermions: Half integer spins concerning the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions in even dimensional spaces ($d = 2(2n+1)$ or $d = 4n$), as discussed in Subsects. (2.2, 2.4) (and in App. A in a pedagogical way). With respect to the subgroups of the $SO(d-1,1)$ group the Clifford odd "basis vectors" appear in the fundamental representations, as illustrated in Subsects. 2.3.

In this article, it is demonstrated that Clifford even algebra is offering the description of the internal space of boson fields. The Clifford even "basis vectors" — the superposition of even products of γ^a's — in a tensor product with the basis in ordinary space form the creation and annihilation operators which manifest the commuting properties of the second quantized boson fields, offering the explanation for the second quantization postulates for boson fields [16,15]. The Clifford even "basis vectors" have all the properties of boson fields: Integer spins for the Cartan subalgebra members of the Lorentz algebra in the internal space of bosons, as discussed in Subsects. 2.2.

With respect to the subgroups of the $SO(d-1,1)$ group the Clifford even "basis vectors" manifest the adjoint representations, as illustrated in Subsect. 2.3.

There are two kinds of anti-commuting algebras [6]: The Grassmann algebra, offering in d-dimensional space 2.2^d operators (2^d θ^a's and 2^d $\frac{\partial}{\partial\theta_a}$'s, Hermitian conjugated to each other, Eq. (3)), and the two Clifford subalgebras, each with 2^d operators named γ^a's and $\tilde{\gamma}^a$'s, respectively, [6,10], Eqs. (2)-(6).

The operators in each of the two Clifford subalgebras appear in even-dimensional spaces in two groups of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ of the Clifford odd operators (the odd products of either γ^a's in one subalgebra or of $\tilde{\gamma}^a$'s in the other subalgebra), which are Hermitian conjugated to each other: In each Clifford odd group of any of the two subalgebras, there appear $2^{\frac{d}{2}-1}$ irreducible representation each with the $2^{\frac{d}{2}-1}$ members and the group of their Hermitian conjugated partners.

There are as well the Clifford even operators (the even products of either γ^a's in one subalgebra or of $\tilde{\gamma}^a$'s in another subalgebra) which again appear in two groups of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members each. In the case of the Clifford even objects, the members of each group of $2^{\frac{d}{2}-1}\times 2^{\frac{d}{2}-1}$ members have the Hermitian conjugated partners within the same group, Subsect. 2.2, Table 1.

The Grassmann algebra operators are expressible with the operators of the two Clifford subalgebras and opposite, Eq. (5). The two Clifford sub-algebras are independent of each other, Eq. (6), forming two independent spaces.

Either the Grassmann algebra [12] or the two Clifford subalgebras can be used to describe the internal space of anti-commuting objects, if the superposition of odd products of operators (θ^a's or γ^a's, or $\tilde{\gamma}^a$'s) are used to describe the internal space of these objects. The commuting objects must be a superposition of even products of operators (θ^a's or γ^a's or $\tilde{\gamma}^a$'s).

No integer spin anti-commuting objects have been observed so far, and to describe the internal space of the so far observed fermions only one of the two Clifford odd subalgebras are needed.

The problem can be solved by reducing the two Clifford subalgebras to only one, the one (chosen to be) determined by γ^a's. The decision that $\tilde{\gamma}^a$'s apply on γ^a as follows: $\{\tilde{\gamma}^a B = (-)^B\, i\, B\gamma^a\}\,|\psi_{oc}>$, Eq. (7), (with $(-)^B = -1$, if B is a function of odd products of γ^a's, otherwise $(-)^B = 1$) enables that $2^{\frac{d}{2}-1}$ irreducible representations of $S^{ab} = \frac{i}{2}\,\{\gamma^a\,,\,\gamma^b\}_-$ (each with the $2^{\frac{d}{2}-1}$ members) obtain the family quantum numbers determined by $\tilde{S}^{ab} = \frac{i}{2}\,\{\tilde{\gamma}^a\,,\,\tilde{\gamma}^b\}_-$.

The decision to use in the *spin-charge-family* theory in $d = 2(2n+1)$, $n \geq 3$ ($d \geq (13+1)$ indeed), the superposition of the odd products of the Clifford algebra elements γ^a's to describe the internal space of fermions which interact with gravity only (with the vielbeins, the gauge fields of momenta, and the two kinds of the spin connection fields, the gauge fields of S^{ab} and $\tilde{S}^{ab}$, respectively), Eq. (1), offers not only the explanation for all the assumed properties of fermions and bosons in the *standard model*, with the appearance of the families of quarks and leptons and antiquarks and antileptons ([14] and the references therein) and of the corresponding vector gauge fields and the Higgs's scalars included [11], but also for the appearance of the dark matter [35] in the universe, for the explanation of the matter/antimatter asymmetry in the universe [8], and for several other observed phenomena, making several predictions [7,33–37].

The recognition that the use of the superposition of the even products of the Clifford algebra elements γ^a's to describe the internal space of boson fields, what appears to manifest all the properties of the observed boson fields, as demonstrated in this article, makes clear that the Clifford algebra offers not only the explanation for the postulates of the second quantized anti-commuting fermion fields but also for the postulates of the second quantized boson fields.

This recognition, however, offers the possibility to relate

$$\begin{aligned}
&\{\frac{1}{2}\sum_{ab} S^{ab}\,\omega_{ab\alpha}\}\sum_{m}\beta^{mf}\,\hat{\mathbf{b}}^{m\dagger}_{f}(\vec{p}) \;\text{ to}\,\{\sum_{m'f'} {}^{I}\hat{\mathcal{A}}^{m'\dagger}_{f'}\,{}^{I}\mathcal{C}^{m'}{}_{f`\alpha}\}\sum_{m}\beta^{mf}\,\hat{\mathbf{b}}^{m\dagger}_{f}(\vec{p})\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall f \text{ and} \forall\,\beta^{mf}\,,\\
&\qquad\mathcal{S}^{cd}\sum_{ab}(c^{ab}{}_{mf}\,\omega_{ab\alpha}) \;\text{ to}\,\mathcal{S}^{cd}\,({}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha})\,,\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall\,(m,\,f),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall \text{ Cartan subalgebra member}\,\mathcal{S}^{cd}\,,
\end{aligned}$$

and equivalently for ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}{}_{f\alpha}$ and $\tilde{\omega}_{ab\alpha}$, what offers the possibility to replace the covariant derivative $p_{0\alpha}$

$$p_{0\alpha} = p_\alpha - \frac{1}{2}S^{ab}\omega_{ab\alpha} - \frac{1}{2}\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$$

in Eq. (1) with

$$p_{0\alpha} = p_\alpha - \sum_{mf} {}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{I}\mathcal{C}^{m}{}_{f\alpha} - \sum_{mf} {}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}{}_{f\alpha}\,,$$

where the relations among ${}^{I}\hat{\mathcal{A}}^{m\dagger}_{f}{}^{I}\mathcal{C}^{m}_{f\alpha}$ and ${}^{II}\hat{\mathcal{A}}^{m\dagger}_{f}\,{}^{II}\mathcal{C}^{m}_{f\alpha}$ with respect to $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, not discussed directly in this article, need additional study and explanation.

Although the properties of the Clifford odd and even "basis vectors" and correspondingly of the creation and annihilation operators for fermion and boson fields are, hopefully, demonstrated in this article, yet the proposed way of the second quantization of fields, the fermion and the boson ones needs further study to find out what new can the description of the internal space of fermions and bosons bring into the understanding of the second quantized fields.

This study showing up that the Clifford algebra can be used to describe the internal spaces of fermion and boson fields equivalently, offering correspondingly the explanation for the second quantization postulates for fermion and boson fields is opening a new insight into the quantum field theory, since studies of the interaction of fermion fields with boson fields and of boson fields with boson fields so far look very promising.

The study of properties of the second quantized boson fields, the internal space of which is described by Clifford even algebra has just started and needs further consideration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

The author thanks Department of Physics, FMF, University of Ljubljana, Society of Mathematicians, Physicists and Astronomers of Slovenia, for supporting the research on the *spin-charge-family* theory by offering the room and computer facilities and Matjaž Breskvar of Beyond Semiconductor for donations, in particular for the annual workshops entitled "What comes beyond the standard models".

Appendix A. "Basis vectors" in $d = (3 + 1)$

This section, suggested by the referee, is to illustrate on a simple case of $d = (3 + 1)$ the properties of "basis vectors" when describing internal spaces of fermions and bosons by the Clifford algebra: i. The way of constructing the "basis vectors" for fermions which appear in families and for bosons which have no families. ii. The manifestation of anti-commutativity of the second quantized fermion fields and commutativity of the second quantized boson fields. iii. The creation and annihilation operators, described by a tensor product, $*_T$, of the "basis vectors" and their Hermitian conjugated partners with the basis in ordinary space-time.

This section is a short overview of some sections presented in the article [18], equipped by concrete examples of "basis vectors" for fermions and bosons in $d = (3 + 1)$.

"Basis vectors"

Let us start by arranging the "basis vectors" as a superposition of products of (operators[17]) γ^a, each "basis vector" is the eigenvector of all the Cartan subalgebra members, Eq. (8). To achieve

[17] We repeat that we treat γ^a as operators, not as matrices. We write "basis vectors" as the superposition of products of γ^a. If we want to look for a matrix representation of any operator, say S^{ab}, we arrange the "basis vectors" into a series and write a matrix of transformations caused by the operator. However, we do not need to look for the matrix representations of the operators since we can directly calculate the application of any operators on "basis vectors".

this, we arrange "basis vectors" to be products of nilpotents and projectors, Eqs. (9), (10), so that every nilpotent and every projector is the eigenvector of one of the Cartan subalgebra members.

Example 1. Let us notice that, for example, two nilpotents anti-commute, while one nilpotent and one projector (or two projectors) commute due to Eq. (6):
$\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(\gamma^1-i\gamma^2)=-\frac{1}{2}(\gamma^1-i\gamma^2)\frac{1}{2}(\gamma^0-\gamma^3)$, while $\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)=\frac{1}{2}(1+i\gamma^1\gamma^2)\frac{1}{2}(\gamma^0-\gamma^3)$.

In $d=(3+1)$ there are 16 $(2^{d=4})$ "eigenvectors" of the Cartan subalgebra members (S^{03}, S^{12}) and ($\mathcal{S}^{03}$, $\mathcal{S}^{12}$) of the Lorentz algebras S^{ab} and $\mathcal{S}^{ab}$, Eq. (8).

Half of them are the Clifford odd "basis vectors", appearing in two irreducible representations, in two "families" ($2^{\frac{4}{2}-1}$, $f=(1,2)$), each with two ($2^{\frac{4}{2}-1}$, $m=(1,2)$) members, $\hat{b}^{m\dagger}_f$, and their Hermitian conjugated partners, Eq. (39).

There are $2^{\frac{4}{2}-1}\times 2^{\frac{4}{2}-1}$ (Clifford odd) Hermitian conjugated partners $\hat{b}^m_f=(\hat{b}^{m\dagger}_f)^\dagger$ appearing in a separate group which is not reachable by S^{ab}, Eq. (40).

There are two separate groups of $2^{\frac{4}{2}-1}\times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", ${}^i\mathcal{A}^{m\dagger}_f, i=(I,II)$, the $2^{\frac{4}{2}-1}$ members of each are self-adjoint, the rest have their Hermitian conjugated partners within the same group, Eqs. (42), (43).

All the members of each group are reachable by S^{ab} or $\tilde{S}^{ab}$ from any starting "basis vector" ${}^i\mathcal{A}^{1\dagger}_1$.

Example 2. $\hat{b}^{m=1\dagger}_{f=1}=\stackrel{03}{(+i)}\stackrel{12}{[+]}$ $(=\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2))$ is a Clifford odd "basis vector", its Hermitian conjugated partner, Eq. (6), is $\hat{b}^{m=1}_{f=1}=\stackrel{03}{(-i)}\stackrel{12}{[+]}$ $(=\frac{1}{2}(\gamma^0+\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)$, not reachable by either S^{ab} or by $\tilde{S}^{ab}$ from any of two members in any of two "families" of the group of $\hat{b}^{m\dagger}_f$, presented in Eq. (39).

${}^I\mathcal{A}^{m=1\dagger}_{f=1}(=\stackrel{03}{[+i]}\stackrel{12}{[+]}=\frac{1}{2}(1+\gamma^0\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)$ is self-adjoint, ${}^I\mathcal{A}^{m=2\dagger}_{f=1}(=\stackrel{03}{(-i)}\stackrel{12}{(-)}=\frac{1}{2}(\gamma^0+\gamma^3)(\gamma^1-i\gamma^2)$. Its Hermitian conjugated partner, belonging to the same group, is ${}^I\mathcal{A}^{m=1\dagger}_{f=2}$ and is reachable from ${}^I\mathcal{A}^{m=1\dagger}_{f=1}$ by the application of $\tilde{S}^{01}$, since $\tilde{\gamma}^0 *_A \stackrel{03}{[+i]}=i\stackrel{03}{(+i)}$ and $\tilde{\gamma}^1 *_A \stackrel{12}{[+]}=i\stackrel{12}{(+)}$.

Clifford odd "basis vectors"

Let us first present the Clifford odd anti-commuting "basis vectors", appearing in two "families" $\hat{b}^{m\dagger}_f$, and their Hermitian conjugated partners $(\hat{b}^{m\dagger}_f)^\dagger$. Each member of the two groups is a product of one nilpotent and one projector. We choose the right-handed Clifford odd "basis vectors".[18] Clifford odd "basis vectors" appear in two families, each family has two members.[19]

[18] We could choose the left-handed Clifford odd "basis vectors" by exchanging the role of 'basis vectors" and their Hermitian conjugated partners.

[19] In the case of $d=(1+1)$, we would have one family with one member only, which must be nilpotent.

Let us notice that members of each of two families have the same quantum numbers (S^{03}, S^{12}). They distinguish in "family" quantum numbers ($\tilde{S}^{03}$, $\tilde{S}^{12}$).

$$
\begin{array}{llll}
\quad f=1 & \quad f=2 & & \\
\tilde{S}^{03}=\frac{i}{2}, \tilde{S}^{12}=-\frac{1}{2} & \tilde{S}^{03}=-\frac{i}{2}, \tilde{S}^{12}=\frac{1}{2} & S^{03} & S^{12} \\
\hat{b}^{1\dagger}_{1} = \stackrel{03}{(+i)}\stackrel{12}{[+]} & \hat{b}^{1\dagger}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(+)} & \frac{i}{2} & \frac{1}{2} \\
\hat{b}^{2\dagger}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(-)} & \hat{b}^{2\dagger}_{2} = \stackrel{03}{(-i)}\stackrel{12}{[-]} & -\frac{i}{2} & -\frac{1}{2} \,.
\end{array}
\tag{39}
$$

We find for their Hermitian conjugated partners

$$
\begin{array}{llll}
S^{03}=-\frac{i}{2}, S^{12}=\frac{1}{2} & S^{03}=\frac{i}{2}, S^{12}=-\frac{1}{2} & \tilde{S}^{03} & \tilde{S}^{12} \\
\hat{b}^{1}_{1} = \stackrel{03}{(-i)}\stackrel{12}{[+]} & \hat{b}^{1}_{2} = \stackrel{03}{[+i]}\stackrel{12}{(-)} & -\frac{i}{2} & -\frac{1}{2} \\
\hat{b}^{2}_{1} = \stackrel{03}{[-i]}\stackrel{12}{(+)} & \hat{b}^{2}_{2} = \stackrel{03}{(+i)}\stackrel{12}{[-]} & \frac{i}{2} & \frac{1}{2} \,.
\end{array}
\tag{40}
$$

The vacuum state $|\psi_{oc}>$, Eq. (15), on which the Clifford odd "basis vectors" apply is equal to: $|\psi_{oc}>= \frac{1}{\sqrt{2}}(\stackrel{03}{[-i]}\stackrel{12}{[+]}+\stackrel{03}{[+i]}\stackrel{12}{[-]})$.

Let us recognize that the Clifford odd "basis vectors" anti-commute due to the odd number of nilpotents, Example 1. And they are orthogonal according to Eqs. (47), (48), (49): $\hat{b}^{m\dagger}_{f} *_A \hat{b}^{m'\dagger}_{f'} = 0$.

Example 3. According to the vacuum state presented above, one finds that, for example, $\hat{b}^{1\dagger}_{1}(=\stackrel{03}{(+i)}\stackrel{12}{[+]})|\psi_{oc}>$ is $\hat{b}^{1\dagger}_{1}$ back, since $\stackrel{03}{(+i)}\stackrel{12}{[+]} *_A \stackrel{03}{[-i]}\stackrel{12}{[+]}=\stackrel{03}{(+i)}\stackrel{12}{[+]}$, according to Eq. (47), while $\stackrel{03}{(-i)}\stackrel{12}{[+]} *_A \stackrel{03}{[-i]}\stackrel{12}{[+]}=0$ (due to $(\gamma^0+\gamma^3)(1-\gamma^0\gamma^3)=0$).

Let us apply S^{01} and $\tilde{S}^{01}$ on some of the "basis vectors" $\hat{b}^{m\dagger}_{f}$, say $\hat{b}^{1\dagger}_{1}$.

When applying $S^{01}=\frac{i}{2}\gamma^0\gamma^1$ on $\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)(\equiv\stackrel{03}{(+i)}\stackrel{12}{[+]})$ we get $-\frac{i}{2}\frac{1}{2}(1-\gamma^0\gamma^3)\frac{1}{2}(\gamma^1-i\gamma^2)(\equiv(-\frac{i}{2}\stackrel{03}{[-i]}\stackrel{12}{(-)}))$.

When applying $\tilde{S}^{01}=\frac{i}{2}\tilde{\gamma}^0\tilde{\gamma}^1$ on $\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)(\equiv\stackrel{03}{(+i)}\stackrel{12}{[+]})$ we get, according to Eq. (7), or if using Eq. (11), $-\frac{i}{2}\frac{1}{2}(1+\gamma^0\gamma^3)\frac{1}{2}(\gamma^1+i\gamma^2)(\equiv(-\frac{i}{2}\stackrel{03}{[+i]}\stackrel{12}{(+)}))$.

It then follows, after using Eqs. (11), (47), (48), (49) or just the starting relation, Eq. (6), and taking into account the above concrete evaluations, the relations of Eq. (16) for our particular case

$$
\begin{aligned}
\hat{b}^{m\dagger}_{f} *_A |\psi_{oc}> &= |\psi^{m}_{f}>\,,\\
\hat{b}^{m}_{f} *_A |\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m\dagger}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{-} *_A |\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'}_{f'}\}_{-} *_A |\psi_{oc}> &= 0\cdot|\psi_{oc}>\,,\\
\{\hat{b}^{m}_{f}, \hat{b}^{m'\dagger}_{f'}\}_{-} *_A |\psi_{oc}> &= \delta^{mm'}\delta_{ff'}|\psi_{oc}>\,.
\end{aligned}
\tag{41}
$$

The last relation of Eq. (41) takes into account that each "basis vector" carries the "family" quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra members, Eq. (8), and the appropriate normalization of "basis vectors", Eqs. (39), (40).

Clifford even "basis vectors"

Besides $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford odd "basis vectors" and the same number of their Hermitian conjugated partners, Eqs. (39), (40), the Clifford algebra objects offer two groups of $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ Clifford even "basis vectors", the members of the group ${}^{I}\mathcal{A}^{m\dagger}_{f}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}$, which have Hermitian conjugated partners within the same group or are self-adjoint.[20] We have the group ${}^{I}\mathcal{A}^{m\dagger}_{f}$, $m=(1,2)$, $f=(1,2)$, the members of which are Hermitian conjugated to each other or are self-adjoint,

$$\begin{array}{lcclcc} & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\ {}^{I}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[+]} & 0 & 0 \;, & {}^{I}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(+i)}\stackrel{12}{(+)} & i & 1 \\ {}^{I}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(-)} & -i & -1 \;, & {}^{I}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[-i]}\stackrel{12}{[-]} & 0 & 0\,, \end{array} \tag{42}$$

and the group ${}^{II}\mathcal{A}^{m\dagger}_{f}$, $m=(1,2)$, $f=(1,2)$, the members of which are either Hermitian conjugated to each other or are self adjoint

$$\begin{array}{lcclcc} & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\ {}^{II}\mathcal{A}^{1\dagger}_{1} = \stackrel{03}{[+i]}\stackrel{12}{[-]} & 0 & 0 \;, & {}^{II}\mathcal{A}^{1\dagger}_{2} = \stackrel{03}{(+i)}\stackrel{12}{(-)} & i & -1 \\ {}^{II}\mathcal{A}^{2\dagger}_{1} = \stackrel{03}{(-i)}\stackrel{12}{(+)} & -i & 1 \;, & {}^{II}\mathcal{A}^{2\dagger}_{2} = \stackrel{03}{[-i]}\stackrel{12}{[+]} & 0 & 0\,. \end{array} \tag{43}$$

The Clifford even "basis vectors" have no families. The two groups, ${}^{I}\mathcal{A}^{m\dagger}_{f}$ and ${}^{II}\mathcal{A}^{m\dagger}_{f}$ (they are not reachable from one another by $\mathcal{S}^{ab}$), are orthogonal (which can easily be checked, since $\stackrel{ab}{(\pm k)} *_A \stackrel{ab}{(\pm k)}= 0$, and $\stackrel{ab}{[\pm k]} *_A \stackrel{ab}{[\mp k]}= 0$).

$$ {}^{I}\mathcal{A}^{m\dagger}_{f} *_A {}^{II}\mathcal{A}^{m'\dagger}_{f`} = 0, \quad \text{for any } (m, m', f, f`)\,. \tag{44}$$

Application of ${}^{i}\mathcal{A}^{m\dagger}_{f}, i=(I,II)$ on $\hat{b}^{m\dagger}_{f}$

Let us demonstrate the application of ${}^{i}\mathcal{A}^{m\dagger}_{f}, i=(I,II)$, on the Clifford odd "basis vectors" $\hat{b}^{m\dagger}_{f}$, Eqs. (20), (23), for our particular case $d=(3+1)$ and compare the result with the result of application S^{ab} and $\tilde{S}^{ab}$ on $\hat{b}^{m\dagger}_{f}$ evaluated above in Example 3. We found, for example, that $S^{01}(=\frac{i}{2}\gamma^0\gamma^1) *_A \hat{b}^{1\dagger}_{1}(=\frac{1}{2}(\gamma^0-\gamma^3)\frac{1}{2}(1+i\gamma^1\gamma^2)(=\stackrel{03}{(+i)}\stackrel{12}{[+]}) = -\frac{i}{2}\frac{1}{2}(1-\gamma^0\gamma^3)\frac{1}{2}(\gamma^1-i\gamma^2)(=(-\frac{i}{2}\stackrel{03}{[-i]}\stackrel{12}{(-i)}) = -\frac{i}{2}\hat{b}^{2\dagger}_{1}$.

Applying ${}^{I}\mathcal{A}^{2\dagger}_{1}(=\stackrel{03}{(-i)}\stackrel{12}{(-)}) *_A \hat{b}^{1\dagger}_{1}(=\stackrel{03}{(+i)}\stackrel{12}{[+]}) = -\stackrel{03}{[-i]}\stackrel{12}{(-)}$, which is $-\hat{b}^{2\dagger}_{1}$, presented in Eq. (39). We obtain in both cases the same result, up to the factor $\frac{i}{2}$ (in front of $\gamma^0\gamma^1$ in S^{01}). In the second case one sees that ${}^{I}\mathcal{A}^{2\dagger}_{1}$ (carrying $\mathcal{S}^{03}=-i, \mathcal{S}^{12}=-1$) transfers these quantum numbers to $\hat{b}^{1\dagger}_{1}$ (carrying $S^{03}=\frac{i}{2}, S^{12}=\frac{1}{2}$) what results in $\hat{b}^{2\dagger}_{1}$ (carrying $S^{03}=\frac{-i}{2}, S^{12}=\frac{-1}{2}$).

20 Let be repeated that $\mathcal{S}^{ab} = S^{ab} + \tilde{S}^{ab}$ [15].

We can check what the application of the rest three ${}^{I}\mathcal{A}^{m\dagger}_{f}$, do when applying on $\hat{b}^{m\dagger}_{f}$. The self-adjoint member carrying $\mathcal{S}^{03}=0, \mathcal{S}^{12}=0$, either gives $\hat{b}^{m\dagger}_{f}$ back, or gives zero, according to Eq. (47). The Clifford even "basis vectors", carrying non zero $\mathcal{S}^{03}$ and $\mathcal{S}^{12}$ transfer their internal values to $\hat{b}^{m\dagger}_{f}$ or give zero. In all cases ${}^{I}\mathcal{A}^{m\dagger}_{f}$ transform a "family" member to another or the same "family" member of the same "family".

Example 4. ${}^{I}\mathcal{A}^{1\dagger}_{1}(=\stackrel{03}{[+i]}\stackrel{12}{[+]}) *_A \hat{b}^{1\dagger}_{1}(=\stackrel{03}{(+i)}\stackrel{12}{[+]}) = \hat{b}^{1\dagger}_{1}(=\stackrel{03}{(+i)}\stackrel{12}{[+]})$, ${}^{I}\mathcal{A}^{1\dagger}_{1}(=\stackrel{03}{[+i]}\stackrel{12}{[+]}) *_A \hat{b}^{1\dagger}_{2}(=\stackrel{03}{[+i]}\stackrel{12}{(+)}) = \hat{b}^{1\dagger}_{2}(=\stackrel{03}{[+i]}\stackrel{12}{(+)})$, ${}^{I}\mathcal{A}^{2\dagger}_{1}(=\stackrel{03}{(-i)}\stackrel{12}{(-)}) *_A \hat{b}^{1\dagger}_{2}(=\stackrel{03}{[+i]}\stackrel{12}{(+)}) = -\hat{b}^{2\dagger}_{2}(=\stackrel{03}{(-i)}\stackrel{12}{[-]})$, ${}^{I}\mathcal{A}^{2\dagger}_{1}(=\stackrel{03}{(-i)}\stackrel{12}{(-)}) *_A \hat{b}^{2\dagger}_{2}(=\stackrel{03}{(-i)}\stackrel{12}{[-]}) = 0$.

One easily sees that the application of ${}^{II}\mathcal{A}^{m\dagger}_{f}$ on $\hat{b}^{m'\dagger}_{f'}$ gives zero for all (m, m', f, f') (due to $\stackrel{ab}{[\pm k]} *_A \stackrel{ab}{[\mp k]}=0$, $\stackrel{ab}{[\pm k]} *_A \stackrel{ab}{(\mp k)}=0$, and similar applications).

We realised in Example 3. that the application of $\tilde{S}^{01} = \frac{i}{2}\tilde{\gamma}^0\tilde{\gamma}^1$ on $\hat{b}^{1\dagger}_{1}$ gives $(-\frac{i}{2}\stackrel{03}{[+i]}\stackrel{12}{(+i)}) = -\frac{i}{2}\hat{b}^{1\dagger}_{2}$.

Let us algebraically, $*_A$, apply ${}^{II}\mathcal{A}^{2\dagger}_{1}(=\stackrel{03}{(-i)}\stackrel{12}{(+)})$, with quantum numbers $(\mathcal{S}^{03}, \mathcal{S}^{12}) = (-i, 1)$, from the right hand side the Clifford odd "basis vector" $\hat{b}^{1\dagger}_{1}$. This application causes the transition of $\hat{b}^{1\dagger}_{1}$ (with quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}) = (\frac{i}{2}, -\frac{1}{2})$ (see Eq. (10)) into $\hat{b}^{1\dagger}_{2}$ (with quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}) = (-\frac{i}{2}, \frac{1}{2})$). ${}^{II}\mathcal{A}^{2\dagger}_{1}$ obviously transfers its quantum numbers to Clifford odd "basis vectors", keeping m unchanged, and changing the "family" quantum number: $\hat{b}^{1\dagger}_{1} *_A {}^{II}\mathcal{A}^{2\dagger}_{1} = \hat{b}^{1\dagger}_{2}$.

We can conclude: The internal space of the Clifford even "basis vectors" has properties of the gauge fields of the Clifford odd "basis vectors"; ${}^{I}\mathcal{A}^{m\dagger}_{f}$ transform "family" members of the Clifford odd "basis vectors" among themselves, keeping the "family" quantum number unchanged, ${}^{II}\mathcal{A}^{m\dagger}_{f}$ transform a particular "family" member into the same "family" member of another "family".

Creation and annihilation operators

To define creation and annihilation operators for fermion and boson fields, we must include besides the internal space, the ordinary space, presented in Eq. (31), which defines the momentum or coordinate part of fermion and boson fields.

We define the creation operators for the single particle fermion states as a tensor product, $*_T$, of the Clifford odd "basis vectors" and the basis in ordinary space, Eq. (33):
$\mathbf{\hat{b}}^{s\dagger}_{f}(\vec{p}) = \sum_m c^{sm}{}_{f}(\vec{p})\,\hat{b}^{\dagger}_{\vec{p}} *_T \hat{b}^{m\dagger}_{f}$. The annihilation operators are their Hermitian conjugated partners.

We have seen in Example 1. that Clifford odd "basis vectors" (having odd products of nilpotents) anti-commute. The commuting objects $\hat{b}^{\dagger}_{\vec{p}}$ (multiplying the "basis vectors") do not change the Clifford oddness of $\mathbf{\hat{b}}^{s\dagger}_{f}(\vec{p})$. The two Clifford odd objects, $\mathbf{\hat{b}}^{s\dagger}_{f}(\vec{p})$ and $\mathbf{\hat{b}}^{s'\dagger}_{f'}(\vec{p}')$, keep their anti-commutativity, fulfilling the anti-commutation relations as presented in Eq. (34). Correspondingly we do not need to postulate anti-commutation relations of Dirac. The Clifford odd "basis vectors" in a tensor product with the basis in ordinary space explain the second quantized postulates for fermion fields.

The Clifford odd "basis vectors" contribute for each $\vec{p}$ a finite number of $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$, the ordinary basis offers infinite possibilities.[21]

Recognizing that internal spaces of fermion fields and their corresponding boson gauge fields are describable in even dimensional spaces by the Clifford odd and even "basis vectors", respectively, it becomes evidently that when including the basis in ordinary space, we must take into account that boson gauge fields have the space index α, which describes the α component of the boson fields in ordinary space.

We multiply, therefore, as presented in Eq. (35), the Clifford even "basis vectors" with the coefficient ${}^{i}\mathcal{C}^{m}{}_{f\alpha}$ carrying the space index α so that the creation operators ${}^{\mathbf{i}}\hat{\mathcal{A}}^{\mathbf{m}\dagger}_{\mathbf{f}\alpha}(\vec{p}) = \hat{b}^{\dagger}_{\vec{p}} *_{T} {}^{i}\mathcal{C}^{m}{}_{f\alpha}\, {}^{i}\hat{\mathcal{A}}^{m\dagger}_{f}$, $i = (I, II)$ carry the space index α.[22] The self-adjoint "basis vectors", like $({}^{i}\hat{\mathcal{A}}^{1\dagger}_{1\alpha}, {}^{i}\hat{\mathcal{A}}^{2\dagger}_{2\alpha}, i = (I, II))$, do not change quantum numbers of the Clifford odd "basis vectors", since they have internal quantum numbers equal to zero.

In higher dimensional space, like in $d = (5 + 1)$, ${}^{I}\hat{\mathcal{A}}^{1\dagger}_{3}$, presented in Table 3, could represent the internal space of a photon field, which transfers to, for example, a fermion and anti-fermion pair with the internal space described by $(\hat{b}^{1\dagger}_{1}, \hat{b}^{3\dagger}_{1})$, presented in Table 2, the momentum in ordinary space.

The subgroup structure of $SU(3)$ gauge fields can be recognized in Fig. 2.

Properties of the gauge fields ${}^{i}\hat{\mathcal{A}}^{m\dagger}_{f\alpha}$ need further studies.

In even dimensional spaces, the Clifford odd and even "basis vectors", describing internal spaces of fermion and boson fields, offer the explanation for the second quantized postulates for fermion and boson fields [15].

Appendix B. Discussion on the open questions of the *standard model* and answers offered by the *spin-charge-family* theory

There are many suggestions in the literature for unifying charges in larger groups, adding additional groups for describing families [1–5], or by going to higher dimensional spaces of the Kaluza-Kline like theories [24–31], what also the *spin-charge-family* is.

Let me present some open questions of the *standard model* and briefly tell the answers offered by the *spin-charge family* theory.

A. Where do fermions — quarks and leptons and antiquarks and antileptons — and their families originate?

The answer offered by the *spin-charge-family* theory: In $d = (13 + 1)$ one irreducible representation of $SO(13, 1)$ analysed with respect to subgroups $SO(7, 1)$ (containing subgroups of $SO(3, 1) \times SU(2) \times SU(2)$) and $SO(6)$ (containing subgroups of $SU(3) \times U(1)$) offers the Clifford odd "basis vectors", describing the internal spaces of quarks and leptons and antiquarks and antileptons, Table 4, as assumed by the *standard model*. The Clifford odd "basis vectors" appear in families.

---

21 An infinitesimally small difference between $\vec{p}$ and $\vec{p'}$ makes two creation operators $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p})$ and $\hat{\mathbf{b}}^{s\dagger}_{f}(\vec{p'})$ with the same "basis vector" describing the internal space of fermion fields still fulfilling the anti-commutation relations (as we learn from atomic physics; two electrons can carry the same spin if they distinguish in the coordinate part of the state).

22 Requiring the local phase symmetry for the fermion part of the action, Eq. (1), would lead to the requirement of the existence of the boson fields with the space index α.

B. Why are charges of quarks so different from charges of leptons, and why have left-handed family members so different charges from the right-handed ones?

The answer offered by the *spin-charge-family* theory: The $SO(7,1)$ part of the "basis vectors" is identical for quarks and leptons and identical for antiquarks and antileptons, Table 4, they distinguish only in the $SU(3)$, the colour or anticolour part, and in the fermion or antifermion $U(1)$ quantum numbers. All families have the same content of $SO(7,1)$, $SU(3)$ and $U(1)$ with respect to S^{ab}. They distinguish only in the family quantum number, determined by $\tilde{S}^{ab}$. The difference between left-handed and right-handed members appears due to the difference in one quantum numbers of the two $SU(2)$ groups, as seen in Table 4.

C. Why do family members — quarks and leptons — manifest such different masses if they all start as massless, as (elegantly) assumed by the *standard model*?

The answer offered by the *spin-charge-family* theory: Masses of quarks and leptons are in this theory determined by the spin connection fields $\omega_{st\sigma}$, the gauge fields of S^{ab},[23] and by $\tilde{\omega}_{st\sigma}$, the gauge fields of $\tilde{S}^{ab}$, which are the same for quarks and leptons.[24] Triplets and singlets are scalar gauge fields with the space index $\sigma = (7,8)$. They have, with respect to the space index, the quantum numbers of the Higgs scalars, Ref. ([14], Table 8, Eq. (110,111)).

D. What is the origin of boson fields, of vector fields which are the gauge fields of fermions, and the Higgs' scalars and the Yukawa couplings? Have all boson fields, with gravity and scalar fields included a common origin?

The answer offered by the *spin-charge-family* theory: In a simple starting action, Eq. (1), boson fields origin in gravity — in vielbeins and two kinds of spin connection fields, $\omega_{ab\alpha}$ and $\tilde{\omega}_{ab\alpha}$, in $d = (13+1)$ — and manifest in $d = (3+1)$ as vector gauge fields, $\alpha = (0,1,2,3)$, or scalar gauge fields, $\alpha \geq 5$ [11], ([14], Sect. 6 and references therein). Boson gauge fields are massless as there are fermion fields. The breaks of the starting symmetry makes some gauge fields massive. This article describes the internal space of boson fields by the Clifford even basis vectors, manifesting as the boson gauge fields of the corresponding fermion fields described by the Clifford odd "basis vectors". The description of the boson fields with the Clifford even "basis vectors" confirms the existence of two kinds of spin connection fields as we see in Sects. 2.2 and 2.3, but also open a door to a new understanding of gravity. According to the starting action, Eq. (1), all gauge fields start in $d \geq (13+1)$ as gravity.

E. How are scalar fields connected with the origin of families? How many scalar fields determine properties of the so far (and others possibly be) observed fermions and of weak bosons?

The answer offered by the *spin-charge-family* theory: The interaction between quarks and leptons and the scalar gauge fields, which at the electroweak brake obtain constant values, causes that quarks and leptons and the weak bosons become massive. There are three singlets, they distinguish among quarks and leptons, and two triplets, they do not distinguish among quarks and leptons, which give masses to the lower four families.[25]

F. Where does the *dark matter* originate?

The answer offered by the *spin-charge-family* theory: The theory predicts two groups of four families at low energy. The stable of the upper four groups are candidates to form the dark matter [35].

G. Where does the "ordinary" matter-antimatter asymmetry originate?

The answer offered by the *spin-charge-family* theory: The theory predicts scalars triplets and antitriplets with the space index $\alpha = (9,10,11,12,13,14)$ [8].

H. How can we understand the second quantized fermion and boson fields?

[23] The three $U(1)$ singlets, the gauge fields of the "fermion" quantum number τ^4, of the hypercharge Y, and of the electromagnetic charge Q, determine the difference in masses of quarks and leptons, presented in Table 4, Ref. ([14], Sect, 6.2.2, Eq. (108)).

[24] The two times two $\widetilde{SU}(2)$ triplets are the same for quarks and leptons, forming two groups of four families. Ref. ([14], Sect, 6.2.2, Eq. (108).

[25] There are the same three singlets and two additional triplets, which determine the masses of the upper four families-explaining the existence of the dark matter.

The answer offered by the *spin-charge-family* theory: The main contribution of this article, Sect. 2, is the description of the internal spaces of fermion and boson fields with the superposition of odd (for fermions) and even (for bosons) products of γ^a. The corresponding creation and annihilation operators, which are tensor, $*_T$, products of (finite number) "basis vectors" and (infinite) basis in ordinary space inherit anti-commutativity or commutativity from the corresponding "basis vectors", explaining the postulates for the second quantized fermion and boson fields.

I. What is the dimension of space? $(3+1)$?, $((d-1)+1)$?, ∞?

The answer offered by the *spin-charge-family* theory: We observe $(3+1)$-dimensional space. In order that one irreducible representation (one family) of the Clifford odd "basis vectors", analysed with respect to subgroups $SO(3,1)\times\, SO(4)\, \times SU(3)\, \times U(1)$ of the group $SO(13,1)$ includes all quarks and leptons and antiquarks and antileptons, the space must have $d \geq (13+1)$. (Since the only "elegantly" acceptable numbers are 0 and ∞, the space-time could be ∞.)

The $SO(10)$ theory [2], for example, unifies the charges of fermions and bosons separately. Analysing $SO(10)$ with respect to the corresponding subgroups, the charges of fermions appear in fundamental representations and bosons in adjoint representations.[26]

There are additional open questions answers of which the *spin-charge-family* theory offers.

The *spin-charge-family* theory has to answer the question common to all the Kaluza-Klein-like theories: How and why the space we observe has $d = (3+1)$ dimensions? The proposed description of the internal spaces of fermion and boson fields might help.

Appendix C. Some useful relations in Grassmann and Clifford algebras, needed also in App. D

This appendix contains the helpful relations needed for the reader of this paper. For more detailed explanations and for proofs, the reader is kindly asked to read [14] and the references therein.

For fermions, the operator of handedness Γ^d is determined as follows:

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}}\gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}\,, & \text{for d even}\,,\\ (i)^{\frac{d-1}{2}}\,, & \text{for d odd}\,. \end{cases} \tag{45}$$

The vacuum state for the Clifford odd "basis vectors", $|\psi_{oc}>$, is defined as

$$|\psi_{oc}>= \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}^m_{f}{}_{*A} \hat{b}^{m\dagger}_f \,|\,1\,>\,. \tag{46}$$

Taking into account that the Clifford objects γ^a and $\tilde{\gamma}^a$ fulfil relations of Eq. (6), one obtains beside the relations presented in Eq. (11) the following ones

$$\begin{gathered}
\stackrel{ab}{(k)}\stackrel{ab}{(-k)} = \eta^{aa}\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{(-k)}\stackrel{ab}{(k)} = \eta^{aa}\stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{(k)}\stackrel{ab}{[-k]} = \stackrel{ab}{(k)}\,,\\
\stackrel{ab}{(-k)}\stackrel{ab}{[k]} = \stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{(-k)} = 0\,, \quad \stackrel{ab}{[k]}\stackrel{ab}{[-k]} = 0\,,\\
\stackrel{ab}{\widetilde{(k)}}\stackrel{ab}{(k)} = 0\,, \quad \stackrel{ab}{\widetilde{(k)}}\stackrel{ab}{(-k)} = -i\eta^{aa}\stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(k)} = -i\eta^{aa}\stackrel{ab}{[k]}\,, \quad \stackrel{ab}{\widetilde{(k)}}\stackrel{ab}{[k]} = i\stackrel{ab}{(k)}\,,
\end{gathered}$$

[26] The space-time is in unifying theories $(3+1)$, consequently they have to relate handedness and charges "by hand" [22], postulate the existence of antiparticles, and the existence of scalar fields, as does the *standard model.*

$$
\begin{aligned}
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[-k]} = 0\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{[-k]} = i\,\stackrel{ab}{(-k)}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(k)} = \stackrel{ab}{(k)}\,,\\
&\stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(-k)} = 0\,, \quad \stackrel{ab}{\widetilde{[k]}}\stackrel{ab}{[k]} = 0\,, \quad \stackrel{ab}{\widetilde{[-k]}}\stackrel{ab}{[k]} = \stackrel{ab}{[k]}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{[-k]} = \stackrel{ab}{[-k]}\;.
\end{aligned}
\tag{47}
$$

The algebraic multiplication among $\stackrel{ab}{(\tilde{k})}$ and $\stackrel{ab}{[\tilde{k}]}$ goes as in the case of $\stackrel{ab}{(k)}$ and $\stackrel{ab}{[k]}$

$$
\begin{aligned}
&\stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\tilde{k}]} = 0\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\tilde{k})} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{(\tilde{k})}\stackrel{ab}{[\widetilde{-k}]} = \stackrel{ab}{(\tilde{k})}\,, \quad \stackrel{ab}{[\tilde{k}]}\stackrel{ab}{(\widetilde{-k})} = 0\,,\\
&\stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{(\tilde{k})} = \eta^{aa}\,\stackrel{ab}{[-k]}\,, \quad \stackrel{ab}{\widetilde{(-k)}}\stackrel{ab}{\widetilde{[-k]}} = 0\,.
\end{aligned}
\tag{48}
$$

One can further find that

$$
\begin{aligned}
&S^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{(k)} = -\frac{i}{2}\eta^{aa}\eta^{cc}\,\stackrel{ab}{[-k]}\stackrel{cd}{[-k]}\,, \quad S^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{[k]} = \frac{i}{2}\,\stackrel{ab}{(-k)}\stackrel{cd}{(-k)}\,,\\
&S^{ac}\,\stackrel{ab}{(k)}\stackrel{cd}{[k]} = -\frac{i}{2}\eta^{aa}\,\stackrel{ab}{[-k]}\stackrel{cd}{(-k)}\,, \quad S^{ac}\,\stackrel{ab}{[k]}\stackrel{cd}{(k)} = \frac{i}{2}\eta^{cc}\,\stackrel{ab}{(-k)}\stackrel{cd}{[-k]}\;.
\end{aligned}
\tag{49}
$$

Appendix D. One family representation of Clifford odd "basis vectors" in $d = (13 + 1)$

This appendix, is following App. D of Ref. [18], with a short comment on the corresponding gauge vector and scalar fields and fermion and boson representations in $d = (14+1)$-dimensional space included.

In even dimensional space $d = (13 + 1)$ ([15], App. A), one irreducible representation of the Clifford odd "basis vectors", analysed from the point of view of the subgroups $SO(3, 1) \times SO(4)$ (included in $SO(7, 1)$) and $SO(7, 1) \times SO(6)$ (included in $SO(13, 1)$, while $SO(6)$ breaks into $SU(3) \times U(1)$), contains the Clifford odd "basis vectors" describing internal spaces of quarks and leptons and antiquarks, and antileptons with the quantum numbers assumed by the *standard model* before the electroweak break. Since $SO(4)$ contains two $SU(2)$ groups, $Y = \tau^{23} + \tau^4$, one irreducible representation includes the right-handed neutrinos and the left-handed antineutrinos, which are not in the *standard model* scheme.

The Clifford even "basis vectors", analysed to the same subgroups, offer the description of the internal spaces of the corresponding vector and scalar fields, appearing in the *standard model* before the electroweak break [16,15]; as explained in Subsect. 2.2.1.

For an overview of the properties of the vector and scalar gauge fields in the *spin-charge-family* theory, the reader is invited to see Refs. ([14,11] and the references therein). The vector gauge fields, expressed as the superposition of spin connections and vielbeins, carrying the space index $m = (0, 1, 2, 3)$, manifest properties of the observed boson fields. The scalar gauge fields, causing the electroweak break, carry the space index $s = (7, 8)$ and determine the symmetry of mass matrices of quarks and leptons.

In this Table 4, one can check the quantum numbers of the Clifford odd "basis vectors" representing quarks and leptons *and antiquarks and antileptons* if taking into account that all the nilpotents and projectors are eigenvectors of one of the Cartan subalgebra members, $(S^{03}, S^{12}, S^{56}, \ldots, S^{13\,14})$, with the eigenvalues $\pm\frac{i}{2}$ for $\stackrel{ab}{(\pm i)}$ and $\stackrel{ab}{[\pm i]}$, and with the eigenvalues $\pm\frac{1}{2}$ for $\stackrel{ab}{(\pm 1)}$ and $\stackrel{ab}{[\pm 1]}$.

Table 4

The left-handed ($\Gamma^{(13,1)} = -1$, Eq. (45)) irreducible representation of one family of spinors — the product of the odd number of nilpotents and of projectors, which are eigenvectors of the Cartan subalgebra of the $SO(13,1)$ group [8,10], manifesting the subgroup $SO(7,1)$ of the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented. It contains the left-handed ($\Gamma^{(3,1)} = -1$) weak ($SU(2)_I$) charged ($\tau^{13} = \pm\frac{1}{2}$), and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$) quarks and leptons, and the right-handed ($\Gamma^{(3,1)} = 1$) weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm\frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm\frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($c^i = (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})$, carrying the "fermion charge" ($\tau^4 = \frac{1}{6}$). The colourless leptons carry the "fermion charge" ($\tau^4 = -\frac{1}{2}$). The same multiplet contains also the left handed weak ($SU(2)_I$) chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right handed weak ($SU(2)_I$) charged and $SU(2)_{II}$ chargeless antiquarks and antileptons. Antiquarks distinguish from antileptons again only in the $SU(3) \times U(1)$ part: Antiquarks are anti-triplets carrying the "fermion charge" ($\tau^4 = -\frac{1}{6}$). The anti-colourless antileptons carry the "fermion charge" ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$.

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
1	u_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
2	u_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	d_R^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	d_R^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	d_L^{c1}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	d_L^{c1}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	u_L^{c1}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
8	u_L^{c1}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	u_R^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	u_R^{c2}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	d_R^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	d_R^{c2}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	d_L^{c2}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	d_L^{c2}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	u_L^{c2}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	u_L^{c2}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{(+)}\ \stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	u_R^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	u_R^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	d_R^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\ \stackrel{11\,12}{[-]}\ \stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$

Table 4 (*continued*)

i		$\vert^a\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)}=(-1)\,1\,,\ \Gamma^{(6)}=(1)\,-1$ of (anti)quarks and (anti)leptons									
20	d_R^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	d_L^{c3}	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	d_L^{c3}	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	u_L^{c3}	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	u_L^{c3}	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	ν_R	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	e_R	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	e_L	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	e_L	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	ν_L	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\bar{d}_L^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\bar{u}_L^{\bar{c}1}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\bar{u}_L^{\bar{c}1}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\bar{d}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\bar{d}_R^{\bar{c}1}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\bar{u}_R^{\bar{c}1}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{(+)}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\bar{d}_L^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\bar{u}_L^{\bar{c}2}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
44	$\bar{u}_L^{\bar{c}2}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\bar{d}_R^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\bar{d}_R^{\bar{c}2}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\vert\vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$

(*continued on next page*)

Table 4 (*continued*)

i		$\vert^{a}\psi_i>$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^{4}	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)\,1\,,\ \Gamma^{(6)} = (1)\,-1$ of (anti)quarks and (anti)leptons									
47	$\bar{u}_R^{\bar{c}2}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\bar{u}_R^{\bar{c}2}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{[-]}\stackrel{13\,14}{(+)}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\bar{d}_L^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\bar{u}_L^{\bar{c}3}$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\bar{u}_L^{\bar{c}3}$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\bar{d}_R^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\bar{d}_R^{\bar{c}3}$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\bar{u}_R^{\bar{c}3}$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\bar{u}_R^{\bar{c}3}$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{(+)}\stackrel{11\,12}{(+)}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\bar{e}_L$	$\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\bar{e}_L$	$\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\bar{\nu}_L$	$-\stackrel{03}{[-i]}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\bar{\nu}_L$	$-\stackrel{03}{(+i)}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\bar{\nu}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\bar{\nu}_R$	$-\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{(-)}\stackrel{78}{(+)}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\bar{e}_R$	$\stackrel{03}{(+i)}\stackrel{12}{[+]}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\bar{e}_R$	$\stackrel{03}{[-i]}\stackrel{12}{(-)}\vert\stackrel{56}{[+]}\stackrel{78}{[-]}\Vert\stackrel{9\,10}{[-]}\stackrel{11\,12}{[-]}\stackrel{13\,14}{[-]}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Taking into account that the third component of the weak charge, $\tau^{13} = \frac{1}{2}(S^{56} - S^{78})$, for the second $SU(2)$ charge, $\tau^{23} = \frac{1}{2}(S^{56} + S^{78})$, for the colour charge [$\tau^{33} = \frac{1}{2}(S^{9\,10} - S^{11\,12})$ and $\tau^{38} = \frac{1}{2\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14})$], for the "fermion charge" $\tau^{4} = -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14})$, for the hyper charge $Y = \tau^{23} + \tau^{4}$, and electromagnetic charge $Q = Y + \tau^{13}$, one reproduces all the quantum numbers of quarks, leptons, and *antiquarks, and antileptons*. One notices that the $SO(7,1)$ part is the same for quarks and leptons and the same for antiquarks and antileptons. Quarks distinguish from leptons only in the colour and "fermion" quantum numbers and antiquarks distinguish from antileptons only in the anti-colour and "anti-fermion" quantum numbers.

In odd dimensional space, $d = (14+1)$, the eigenstates of handedness are the superposition of one irreducible representation of $SO(13,1)$, presented in Table 4, and the one obtained if on each "basis vector" appearing in $SO(13,1)$ the operator $S^{0\,(14+1)}$ applies, Subsect. 2.2.2, Ref. [18].

Let me point out that in addition to the electroweak break of the *standard model* the break at $\geq 10^{16}$ GeV is needed ([14], and references therein). The condensate of the two right-handed neutrinos causes this break (Ref. [14], Table 6); it interacts with all the scalar and vector gauge fields, except the weak, $U(1)$, $SU(3)$ and the gravitational field in $d = (3+1)$, leaving these gauge fields massless up to the electroweak break, when the scalar fields, leaving massless only the electromagnetic, colour and gravitational fields, cause masses of fermions and weak bosons.

The theory predicts two groups of four families: To the lower group of four families, the three so far observed contribute. The theory predicts the symmetry of both groups to be $SU(2) \times SU(2) \times U(1)$, Ref. ([14], Sect. 7.3), which enable to calculate mixing matrices of quarks and leptons for the accurately enough measured 3×3 sub-matrix of the 4×4 unitary matrix. No sterile neutrinos are needed, and no symmetry of the mass matrices must be guessed [36].

In the literature, one finds a lot of papers trying to reproduce mass matrices and measured mixing matrices for quarks and leptons [41–47].

The stable of the upper four families predicted by the *spin-charge-family* theory is a candidate for the dark matter, as discussed in Refs. [35,14]. In the literature, there are several works suggesting candidates for the dark matter and also for matter/antimatter asymmetry [48,49].

References

[1] H. Georgi, in: C.E. Carlson (Ed.), Particles and Fields, A.I.P., 1975; Google Scholar.

[2] H. Fritzsch, P. Minkowski, Ann. Phys. 93 (1975) 193.

[3] J. Pati, A. Salam, Phys. Rev. D 8 (1973) 1240.

[4] H. Georgy, S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438.

[5] Y.M. Cho, J. Math. Phys. 16 (1975) 2029.

[6] N. Mankoč Borštnik, Spinor and vector representations in four dimensional Grassmann space, J. Math. Phys. 34 (1993) 3731–3745;
N. Mankoč Borštnik, Unification of spin and charges in Grassmann space?, arXiv:hep-th/9408002, IJS.TP.94/22; Mod. Phys. Lett. A 10 (7) (1995) 587–595.

[7] A. Borštnik Bračič, N.S. Mankoč Borštnik, On the origin of families of fermions and their mass matrices, arXiv: hep-ph/0512062, Phys. Rev. D 74 (2006) 073013.

[8] N.S. Mankoč Borštnik, Matter-antimatter asymmetry in the spin-charge-family theory, Phys. Rev. D 91 (2015) 065004, arXiv:1409.7791.

[9] N.S. Mankoč Borštnik, The spin-charge-family theory is explaining the origin of families, of the Higgs and the Yukawa couplings, J. Mod. Phys. 4 (2013) 823, arXiv:1312.1542.

[10] N.S. Mankoč Borštnik, H.B.F. Nielsen, J. Math. Phys. 43 (2002) 5782, arXiv:hep-th/0111257;
N.S. Mankoč Borštnik, H.B.F. Nielsen, How to generate families of spinors, J. Math. Phys. 44 (2003) 4817, arXiv: hep-th/0303224.

[11] N.S. Mankoč Borštnik, D. Lukman, Vector and scalar gauge fields with respect to $d = (3+1)$ in Kaluza-Klein theories and in the spin-charge-family theory, Eur. Phys. J. C 77 (2017) 231.

[12] N.S. Mankoč Borštnik, H.B. Nielsen, Why nature made a choice of Clifford and not Grassmann coordinates, in: N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman (Eds.), Proceedings to the 20^{th} Workshop "What Comes Beyond the Standard Models", Bled, 9-17 of July, 2017, DMFA Založništvo, Ljubljana, December 2017, pp. 89–120, arXiv: 1802.05554v1.

[13] N.S. Mankoč Borštnik, H.B.F. Nielsen, Understanding the second quantization of fermions in Clifford and in Grassmann space, arXiv:2007.03517;
N.S. Mankoč Borštnik, H.B.F. Nielsen, New way of second quantization of fermions — Part I and Part II proceedings, arXiv:2007.03516;
N.S. Mankoč Borštnik, H.B.F. Nielsen, New way of second quantized theory of fermions with either Clifford or Grassmann coordinates and spin-charge-family theory, arXiv:1802.05554v4, arXiv:1902.10628.

[14] N.S. Mankoč Borštnik, H.B. Nielsen, How Does Clifford Algebra Show the Way to the Second Quantized Fermions with Unified Spins, Charges and Families, and with Vector and Scalar Gauge Fields Beyond the Standard Model, Progress in Particle and Nuclear Physics, 2023.

[15] N.S. Mankoč Borštnik, How Clifford algebra can help understand second quantization of fermion and boson fields, arXiv:2210.06256v1 [physics.gen-ph].
[16] N.S. Mankoč Borštnik, How do Clifford algebras show the way to the second quantized fermions with unified spins, charges and families, and to the corresponding second quantized vector and scalar gauge field, in: N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, A. Kleppe (Eds.), Proceedings to the 24^{rd} Workshop "What Comes Beyond the Standard Models", 5 - 11 of July, 2021, DMFA Založništvo, Ljubljana, December 2021, arXiv:2112.04378.
[17] N.S. Mankoč Borštnik, Clifford odd and even objects offer description of internal space of fermions and bosons, respectively, opening new insight into the second quantization of fields, in: The 13^{th} Bienal Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields IARD 2022, Prague, 6 - 9 June, 2023, http://arxiv.org/abs/2210.07004, J. Phys. Conf. Ser., vol. 2482/2023.
[18] N.S. Mankoč Borštnik, Clifford odd and even objects offer description of internal spaces of fermions in even and odd dimensional spaces, arXiv:2301.04466, Symmetry 15 (4) (2023) 818, https://doi.org/10.3390/sym15040818.
[19] P.A.M. Dirac, Proc. R. Soc. Lond. A 117 (1928) 610.
[20] H.A. Bethe, R.W. Jackiw, Intermediate Quantum Mechanics, W.A. Benjamin, New York, 1968.
[21] S. Weinberg, The Quantum Theory of Fields, Cambridge University Press, Cambridge, 2015.
[22] N.S. Mankoč Borštnik, H.B.F. Nielsen, The spin-charge-family theory offers understanding of the triangle anomalies cancellation in the standard model, Fortschr. Phys. (Progress of Physics) (2017) 1700046.
[23] D. Lukman, N.S. Mankoč Borštnik, H.B. Nielsen, An effective two dimensionality cases bring a new hope to the Kaluza-Klein-like theories, New J. Phys. 13 (2011) 103027.
[24] T. Kaluza, On the unification problem in Physics, Sitz.ber. Berl. Acad. (1918) 204;
O. Klein, Quantum theory and five-dimensional relativity, Z. Phys. 37 (1926) 895.
[25] E. Witten, Search for realistic Kaluza-Klein theory, Nucl. Phys. B 186 (1981) 412.
[26] M. Duff, B. Nilsson, C. Pope, Phys. Rep. C 130 (1984) 1;
M. Duff, B. Nilsson, C. Pope, N. Warner, Phys. Lett. B 149 (1984) 60.
[27] T. Appelquist, H.C. Cheng, B.A. Dobrescu, Phys. Rev. D 64 (2001) 035002.
[28] M. Saposhnikov, P. Tinyakov, Phys. Lett. B 515 (2001) 442, arXiv:hep-th/0102161v2, 2001.
[29] C. Wetterich, Nucl. Phys. B 253 (1985) 366.
[30] The authors of the works presented H.C. Lee (Ed.), An Introduction to Kaluza-Klein Theories, World Scientific, Singapore, 1983.
[31] M. Blagojević, Gravitation and Gauge Symmetries, IoP Publishing, Bristol, 2002.
[32] M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, On the origin of families of fermions and their mass matrices — approximate analyses of properties of four families within approach unifying spins and charges, in: Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman (Eds.), Proceedings to the 9^{th} Workshop "What Comes Beyond the Standard Models", Bled, Sept. 16 - 26, 2006, DMFA Založništvo, Ljubljana, December 2006, pp. 25–50, arXiv:hep-ph/0612250.
[33] G. Bregar, M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, Families of quarks and leptons and their mass matrices, in: Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman (Eds.), Proceedings to the 10^{th} International Workshop "What Comes Beyond the Standard Model", 17 -27 of July, 2007, DMFA Založništvo, Ljubljana, December 2007, pp. 53–70, arXiv:0711.4681 [hep-ph].
[34] G. Bregar, M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, Predictions for four families by the approach unifying spins and charges, New J. Phys. 10 (2008) 093002, arXiv:hep-ph/0606159, arXiv:0708.2846 [hep-ph].
[35] G. Bregar, N.S. Mankoč Borštnik, Does dark matter consist of baryons of new stable family quarks?, Phys. Rev. D 80 (2009) 083534.
[36] G. Bregar, N.S. Mankoč Borštnik, Can we predict the fourth family masses for quarks and leptons?, in: N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman (Eds.), Proceedings (arxiv:1403.4441) to the 16th Workshop "What Comes Beyond the Standard Models", Bled, 14-21 of July, 2013, DMFA Založništvo, Ljubljana, December 2013, pp. 31–51, http://arxiv.org/abs/1212.4055.
[37] G. Bregar, N.S. Mankoč Borštnik, The new experimental data for the quarks mixing matrix are in better agreement with the spin-charge-family theory predictions, in: N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman (Eds.), Proceedings to the $17^t h$ Workshop "What Comes Beyond the Standard Models", Bled, 20-28 of July, 2014, DMFA Založništvo, Ljubljana, December 2014, pp. 20–45, arXiv:1502.06786v1.
[38] N.S. Mankoč Borštnik, M. Rosina, Are superheavy stable quark clusters viable candidates for the dark matter?, Int. J. Mod. Phys. D 24 (13) (2015) 1545003.
[39] M. Pavšič, The Landscape of Theoretical Physics: Global View, van der Merwe, A. (Ed.), Kluwer Academic Publishers, New York, NY, USA, 2001.
[40] M. Pavšič, Quantized fields á la Clifford and unification, arXiv:1707.05695, 2017.

[41] H. Fritzsch, Weak-interaction mixing in the six-quark theory, Phys. Lett. B 73 (1978) 317.
[42] C.D. Frogatt, H.B. Nielsen, Hierarchy of quark masses, cabibbo angles and CP violation, Nucl. Phys. B 147 (1979) 277.
[43] C. Jarlskog, Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP nonconservation, Phys. Rev. Lett. 55 (1985) 1039.
[44] C. Gustavo, C. Branco, L. Lavoura, Ansatz for the quark mass matrices allowing for a high top-quark mass, Phys. Rev. D 44 (1991) R582(R).
[45] H. Harari, Y. Nir, B-anti B mixing and relations among quark masses, angles and phases, Phys. Lett. B 195 (1987) 586.
[46] B. Stech, Are the neutrino masses and mixings closely related to the masses and mixings of quarks?, Phys. Lett. B 403 (1997) 114.
[47] G. Altarelli, F. Feruglio, Models of neutrino masses and mixings, New J. Phys. 6 (2004) 106.
[48] R.N. Mohapatra, A unified solution to the big problems of the standard model, arXiv:2207.10619.
[49] P. Frampton, Predictions of additional baryons and mesons, arXiv:2209.05349, 2022, arXiv:2211.09579.