A Transformation Groupoid and Its Representation — A Theory of Dimensionality

Euich Miztani
JEIN Institute of Fundamental Science
euichi@gmail.com

In memory of the late

Prof. Ichiro Yokota

of Shinshu University

and

known for his research of cellular decompositions of classical Lie groups and realizations of exceptional Lie groups.

Abstract

In higher dimensional physics there are usually two ways of dimensional reduction. One is by Kaluza-Klein theory and another by braneworld. In this talk we would like to discuss a third way of dimensional reduction. It is remarkably succinct, integrated by the groupoid and represented by the operation. Additionally, since it has a symmetry, it suggests an unknown conservation law based on Noether's theorem.

Prologue: The Motive and Background

- Since the dawn of modern cosmology (early 20th century), the 3-dimensional sphere or hyperboloid has been the model for the universe we live in. Naturally, the Poincaré Conjecture was not proven at that time.
- In 1921-26, T. Kaluza and O. Klein proposed that gravity and electromagnetic forces can be unified by adding one extra dimension of space to the 3-dimensional space-time (4-dimensional space plus time) called 5-dimensional space-time (later shown to be insufficient). The extra 1-dimensional space, which we cannot perceive, is confined within the subatomic particles as an extremely small closed space. This is generally known as the Kaluza-Klein theory (KK theory), which is one of the essential theories in string theory today.
- Although more complicated Calabi-Yau manifolds are used today, the essential idea is the same as the KK theory. Moreover, 10^{500} different universes are produced from this theory (multiverses).
- Experiments have been conducted to search for this compactified extra dimension, but they have yet to be verified. None of the particle physicists have been able to answer why space of more than three dimensions is compactly wound up in the first place.
- Additionally, such an embedding from higher dimensional space to lower dimensional space should be diffeomorphism. However, they are really diffeomorphism?

・An idea called D-brain (Dirichlet membrane: oscillations of string particles due to Dirichlet boundary conditions) and braneworld are becoming mainstream in string theory. Those ideas are that the extra dimensions are not compactified, but that our universe is a 3-dimensional space (4-dimensional space-time) floating within higher dimensional space.

・If this is the case, it is not surprising that there are subatomic particles as well that are eternally moving in a 2-dimensional plane within 'our 3-dimensional space’. No such strange subatomic particles have yet been discovered.

Simple question:

Could we not discuss this in a simpler way or model?

Occam’s Razor:

The theory or law to be explained should be relatively simple,

or

we should not use more assumptions than necessary to explain them.

Should we not think smarter, based on such a philosophy?

Historical examples

・Epicycles became more and more complicated to correct them as discrepancies with observational facts were found, replacing Copernicus‘ geocentric theory. (Initially, the celestial motion theory with epicycles was more accurate.)

See https://youtu.be/erqsxNFOw4I

・The Lorentz-Fitzgerald contraction, which is a contraction of the ether, was used to explain the medium of electromagnetic waves.

The contraction is a strange hypothesis that space shrinks as the speed of an object increases. Finally, Einstein's special theory of relativity buried all such strange assumptions.

History of spatial dimensions

- Euclid: Definition of points, lines, planes.
- Aristotle: The 3-dimensional volume is 'perfect' and there are no dimensions beyond the third dimension in his celestial theory.
- R. Descartes, P. Fermat: Co-ordinate geometry.
- B. Riemann: In Riemannian geometry, he introduced the line element (an extension of the Pythagorean theorem), which made it possible for the first time to mathematically discuss spaces of four or more dimensions.
- D. Hilbert: Based on the orthogonality of vectors, n-dimensional space is a space in which any number of base vectors are orthogonal to each other and the norm is defined. It is called a Hilbert space.

These ideas assume that low-dimensional space is a subspace of higher (or same) dimensional space. However, is this assumption correct in the strict sense? That will be obviously doubtful, especially when we look at the strangeness of extra-dimensional (higher-dimensional) space in modern physics.

Chapter 1: Concrete Insights

What is the spatial dimension? First of all, let us delve deeper into this matter.

Let us consider again the spatial dimension from the low-dimensional case

・Let us consider the relationship between co-ordinate geometry and its degree of freedom.

Degree of Freedom

- The simplest explanation is that of the direction from point A to point B in the 2-dimensional space.
- We empirically think that point B is only one arbitrary point (as in Cartesian co-ordinates).
- However, is this absolutely true?
- This is actually a special case, because the degree of freedom of the point means moving from A to B. That is, the direction towards point B is completely guaranteed.
- Suppose then, that the point from A to B has no degree of freedom: indecisive to one direction.
- How would that point move?

Point not given a degree of freedom (the 2-dimensional space)

- A point that is not given a degree of freedom cannot determine a single direction.
- In other words, as shown in Fig. 1, the point will move the 'whole event' directions (the entire 360°).

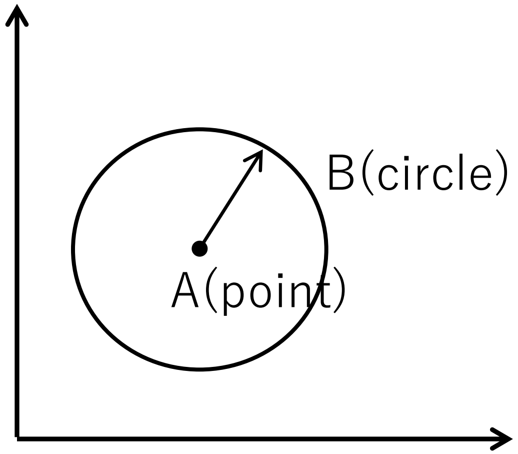


Fig. 1

- This appears to be a wave (pulse wave) propagating in space with no medium. What does this mean?

Here are some things to keep in mind

- The original 1-dimensional space is not a subspace of the 2-dimensional space. A subspace is only a part of the 2-dimensional space. If it is a 1-dimensional straight line as a subspace, the degree of freedom has 'already' been determined (e.g. from point A to point B).
- Now consider the degree of freedom or direction. If a point has 2 degrees of freedom in the 2-dimensional space, it is a point in the 2-dimensional space.
- On the other hand, what does it mean if a point has only one degree of freedom in the 2-dimensional space?

Conclusion: What does it mean that a point is in the 2-dimensional space with only one degree of freedom?

A point has only one degree of freedom in the 2-dimensional space:

- If the point originally existed in the 1-dimensional space with only one degree of freedom and it is moved to the 2-dimensional space, then the point keeps having only one degree of freedom in the 2-dimensional space.
- Now consider the degree of freedom as a stochastic event: the fact that a point moving in the 2-dimensional space has two degrees of freedom means that the point can go in any direction in 360° . On the other hand, if a point has only one degree of freedom in the 2-dimensional space, it cannot arbitrarily decide in which direction to go. Therefore, the point can only move forward the 'whole event' directions. Thus, the point can only move in the 2-dimensional space like a pulse wave.

Let us consider the same thing in the 2-dimensional Cartesian co-ordinate system

- In the ordinary Cartesian co-ordinates, there are two degrees of freedom (x, y).
- If there is only one degree of freedom, then there is only x (or y).
- For example, if x has a value, it can be shown as x=2. This is a 'point' located at 2 in the 1-dimensional space (the number line), but if we consider this in the 2-dimensional space, of course we do not consider it to be (2, 0). Note that by considering y=0, we have given y a degree of freedom. Strictly speaking, we consider this x=2 to be a 'straight line' parallel to the y-axis and passing through 2 on the x-axis.
- If we consider this in the context mentioned above, a point moved from the 1-dimensional space to the 2-dimensional space has no degree of freedom on the y-axis, so it occupies all points on the y-axis (i.e. the 'whole event' in probability theory). Therefore, x=2 can be interpreted as meaning that x=2 is a straight line. It makes sense also in conventional mathematics.

Furthermore, consider x=2 in the 3-dimensional space

Since x=2 has only one degree of freedom in the

3-dimensional space, it can be regarded as a plane parallel to the y-z plane by the same consideration.

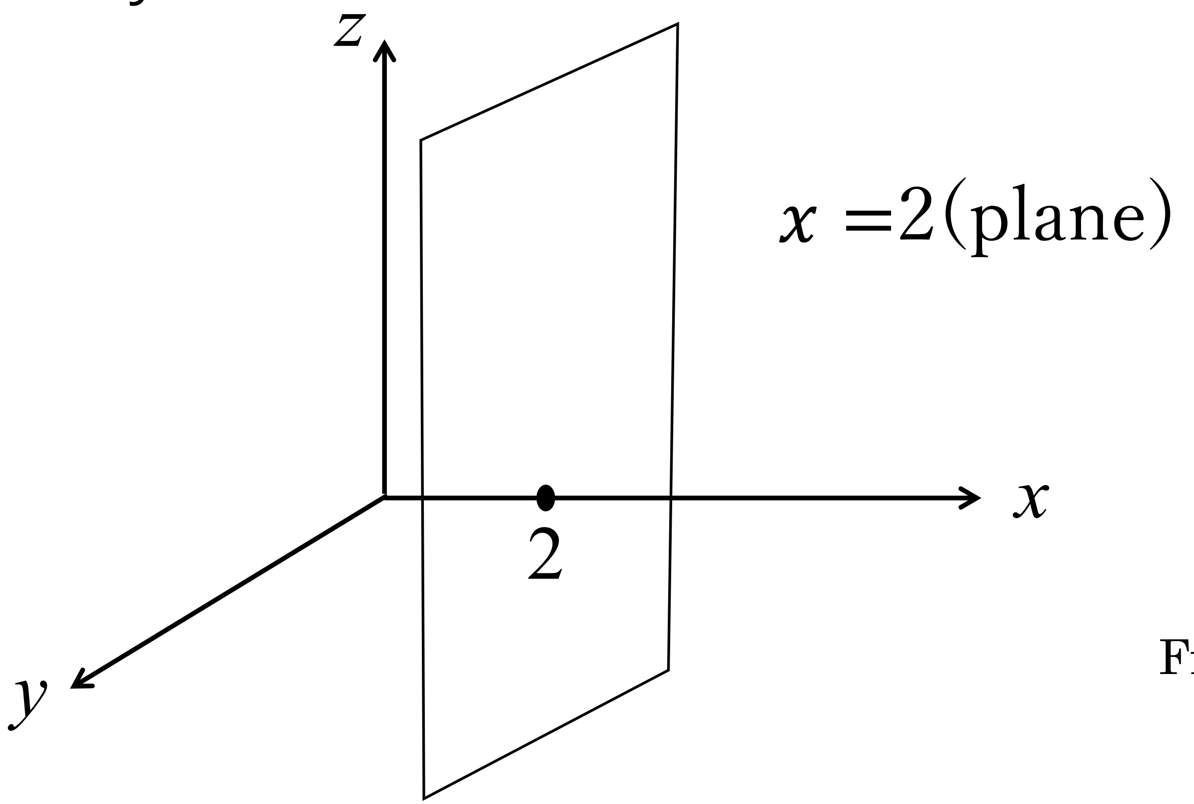


Fig. 2

Now consider a point with one degree of freedom that is transferred from the 1-dimensional space to higher dimensional space and then the first example which spreads out in concentric circles as described earlier:

- This is considered to be a case in the polar co-ordinates.
- Therefore, if the point x=2 is transferred to the 2-dimensional space, it is a circle of radius 2.
- If the point is transferred to the 3-dimensional space, it is a sphere of radius 2.

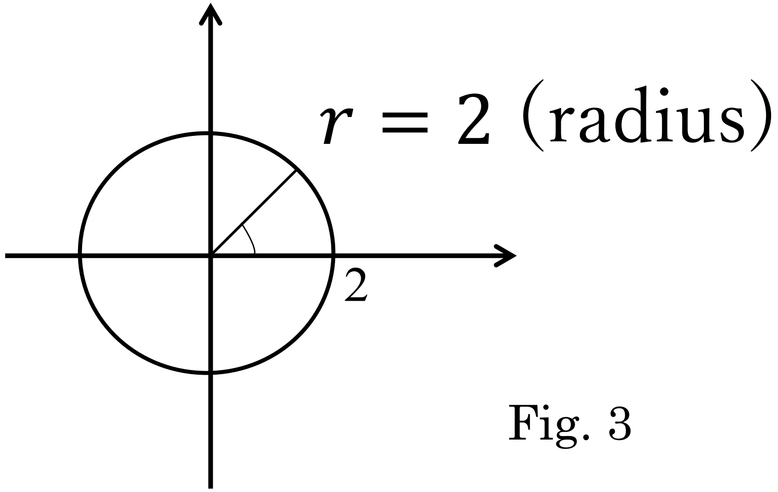


Fig. 3

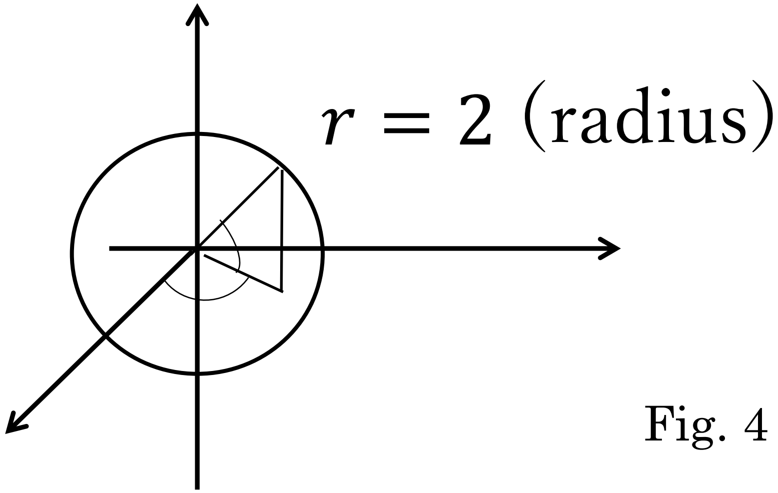


Fig. 4

So far, we have considered points that are transferred from lower dimensional space to higher dimensional space. How does the other way around go?

- The best example of this is the relationship between the equations of a circle and a surface in the homogeneous co-ordinates. For example, the equation of a paraboloid $x^2+y^2=z^2$ is, by algebraic manipulation, the equation of a circle $(x/y)^2+(y/z)^2=X^2+Y^2=1$, shown in Fig. 5 and Fig. 6.
- This can be said to be a projection of a surface in the 3-dimensional space on to a circle in the 2-dimensional space. In other words, it is a transfer from higher dimensional space to lower dimensional space.
- From now on, following the term of the homogenous co-ordinates, the movement from m-dimensional space to n-dimensional space is referred to as ‘projection’.

The projection from the 3-dimensional space to the 2-dimensional space in the homogenous co-ordinates:

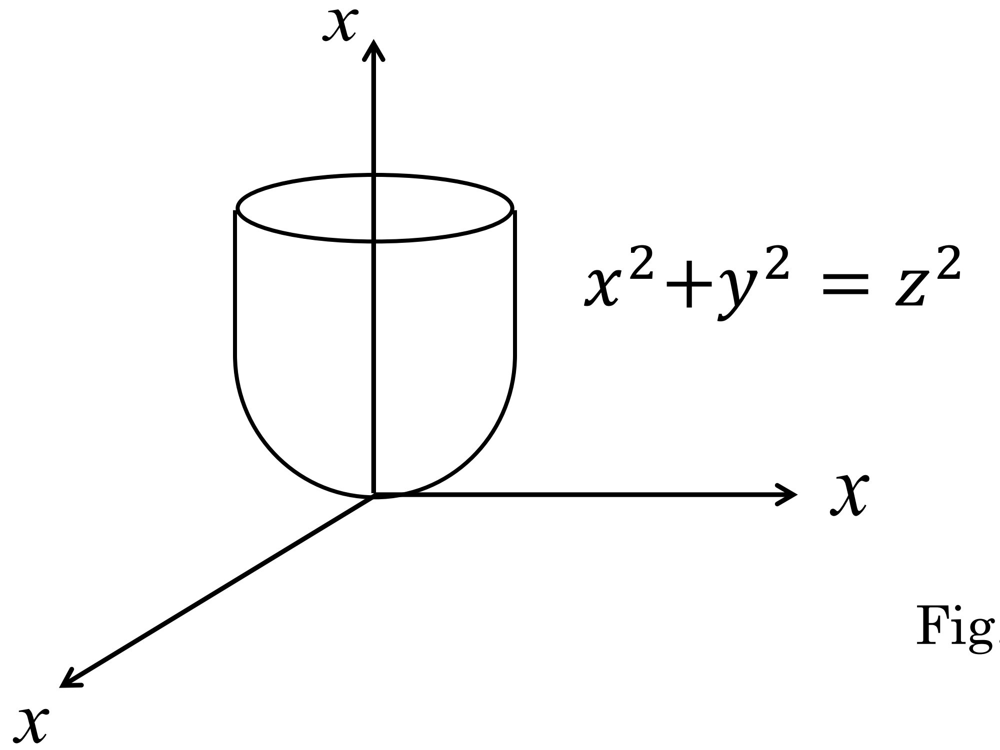


Fig. 5

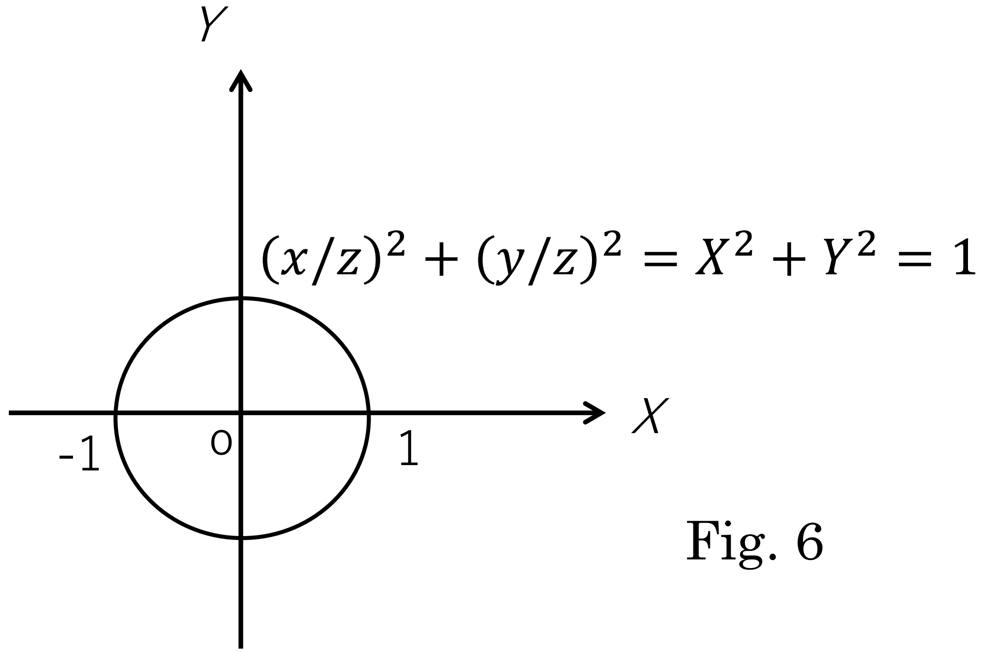


Fig. 6

Finally for this chapter, let us end the discussion above as follows:

- Lower dimensional space is not a subspace of higher dimensional space. They are disjoint each other.
- Any point in the n-dimensional space has n-degrees of freedom: it has n-variables.
- If a point is transferred to a different dimensional space, the number of variables or the degree of freedom never changes.

Chapter 2: Description with Matrices

Let us introduce a specific matrix operator to project a point between mutual dimensions. This matrix is different from a conventional one; it includes a special operator needing a temporary variable for operation, because the number of variables of a point before and after this operation is different. Demonstrating it with attention to the fact, for example, operating by an operator E_{12} to project a point A_1 in the 1-dimensional space into the 2-dimensional space (A_2), the equation is $A_2 = E_{12}A_1$,

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a \\ T \end{pmatrix} = \begin{pmatrix} a \\ DT \end{pmatrix}, \dots (2.1)$$

where D is a matrix element making the dimension higher and T a temporary variable to correspond to the 2-dimensional space after operation. Therefore, DT denotes all real numbers of y at once. This process of Eq. 2.1 is shown in Fig. 7.

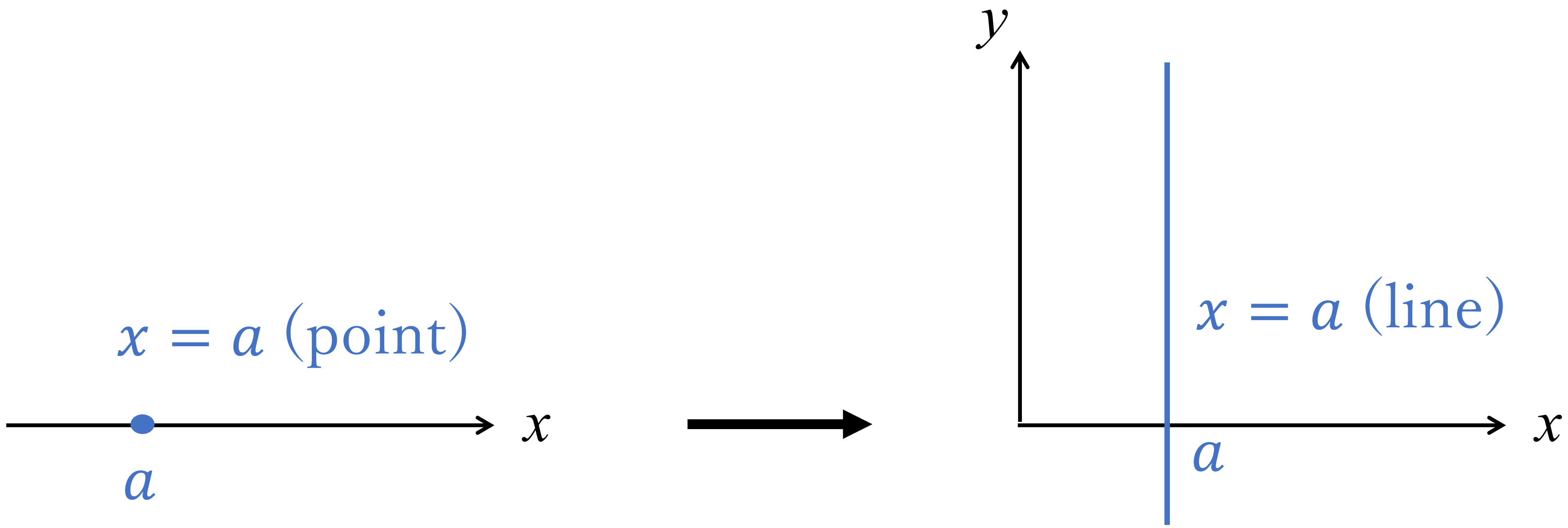


Fig. 7

Operating another case from the 1-dimensional space to the 3-dimensional space, then:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} a \\ T \\ T \end{pmatrix} = \begin{pmatrix} a \\ DT \\ DT \end{pmatrix}. \quad \ldots (2.2)$$

Similarly, projection from the 3-dimensional space to the 2-dimensional space is operated as:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ D^{-1}c \end{pmatrix}. \quad \ldots (2.3)$$

D^{-1} denotes an element making the dimension lower and inverse of D. Eq. 2.3 is shown in Fig. 8. Then, if returning the point projected from the 3-dimensional space into the 2-dimensional space by Eq. 2.3 to the original dimensional space, the operation is as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} a \\ b \\ D^{-1}c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \quad \ldots (2.4)$$

$$\therefore E_{32}E_{23}(= (E_{23})^{-1}E_{23}) = E_{33} \equiv 1. \quad \ldots (2.5)$$

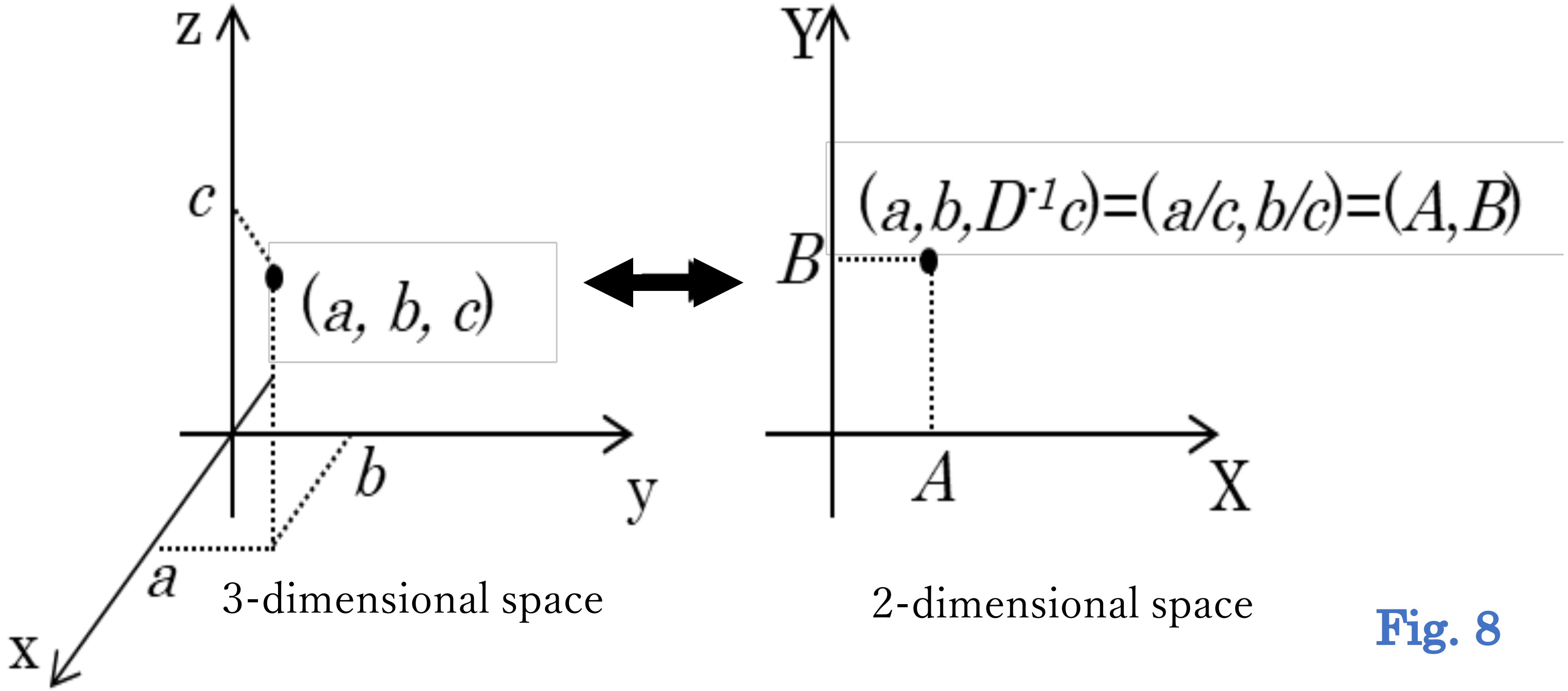

z
c
(a, b, c)
y
b
a
x
3-dimensional space
Y
$(a, b, D^{-1}c) = (a/c, b/c) = (A, B)$
B
A
X
2-dimensional space

Fig. 8

The general operator which is dimensional unit matrix E_{lm} is:

If $l < m$,

$$E_{lm} = diag(\overbrace{1,1,1,\dots,1,1,1}^{l}, \overbrace{D,D,D,\dots,D,D,D}^{m-l}).$$

If $m < l$,

$$E_{lm} = (E_{ml})^{-1} = diag(\overbrace{1,1,1,\dots,1,1,1}^{m}, \overbrace{D^{-1},D^{-1},D^{-1},\dots,D^{-1},D^{-1},D^{-1}}^{l-m}).$$

$$\therefore E_{jk}E_{kj} = E_{jj} \equiv 1 \equiv E_{kj}E_{jk} = E_{kk}. \quad \dots(2\text{-}6)$$

Furthermore,

$$E_{0n} = diag(\overbrace{D,D,D,\dots,D,D,D}^{n}),$$

$$E_{n0} = (E_{0n})^{-1} = diag(\overbrace{D^{-1},D^{-1},D^{-1},\dots,D^{-1},D^{-1},D^{-1}}^{n}).$$

Note: This is not the best example though, the case below is in a series of projections from higher dimensional space to lower dimensional space:

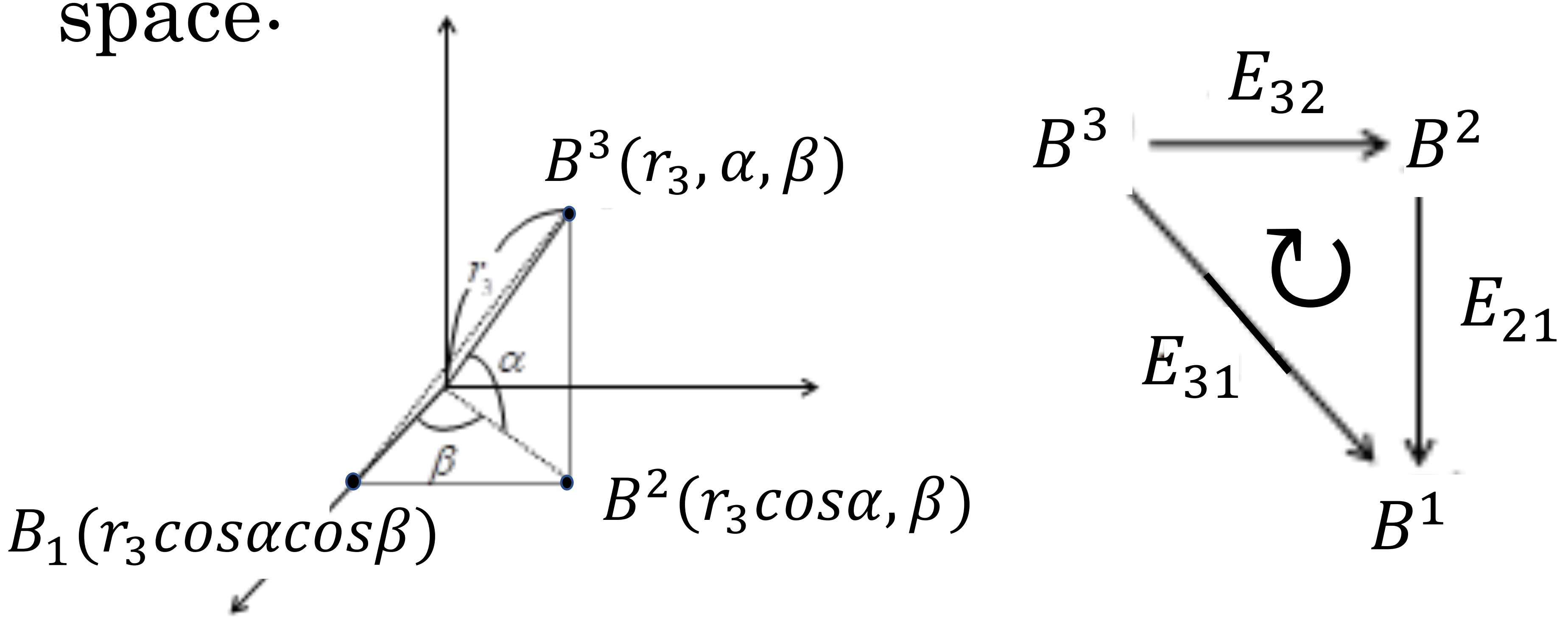


Fig. 9

Chapter 3: The Groupoid

Proposition. *In the former chapter, matrices in a series of operations (partially functional) make the action groupoid. They are indicated by equations as follows*：

i. $E_{lm}E_{mn} = E_{ln}$ (automorphism besides closure, proven in Chap. 4), … (3.1)

ii. $(E_{kl}E_{lm})E_{mn} = E_{kl}(E_{lm}E_{mn})$ (associative), … (3.2)

iii. $E_{jk}E_{kj} = E_{jj} \equiv 1 \equiv E_{kj}E_{jk} = E_{kk}$ (inverse), … (3.3)

Or $E_{lm}E_{ml} = E_{lm}(E_{lm})^{-1} = (E_{ml})^{-1}E_{ml} \equiv 1$ (identity), … (3.4)

iv. $E_{kl}E_{lm}E_{ml} = E_{kl}E_{lm}\ (E_{lm})^{-1} = E_{kl}$ (right identity), … (3.5)

and $E_{lk}E_{kl}E_{lm} = (E_{kl})^{-1}E_{kl}E_{lm} = E_{lm}$ (left identity), … (3.6)

v. $(E_{lm}E_{mn})^{-1} = (E_{mn})^{-1}(E_{lm})^{-1}$, … (3.7)

vi. $E_{jj} \equiv 1$ (identity equivalent to the scalar value). … (3.8)

(i) and (vi) are peculiar to the groupoid.

The operators in chap. 2 explicitly show the groupoid mentioned above. However, we have never calculated such matrices. Therefore, we need to check and verify they really work.

Proof. At first, of the formula (i) is as follows:

a1) If $0 < l < m < n$ (projecting into higher dimensions),

$$x' = E_{lm}x = diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,T}^{m-l})^T$$

$$\therefore\ E_{mn}x' = diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,D,\dots,D}^{n-m})(x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l}\,\overbrace{T,T,\dots,T}^{n-m})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,T}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

a2) If $0 < l < n < m$ (projecting into higher dimensions),

$$x' = E_{lm}x = diag(\overbrace{1,1,\ldots,1}^{l}, \overbrace{D,D,\ldots,D}^{m-l})(x_1, x_2, \ldots, x_l, \overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1, x_2, \ldots, x_l, \overbrace{DT, DT, \ldots, T}^{m-l})^T$$

$$\therefore\ E_{mn}x' = (E_{nm})^{-1}\ x'$$

$$= diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n})(x_1, x_2, \ldots, x_l, \overbrace{DT, DT, \ldots, DT}^{m-l}\ \overbrace{T,T,\ldots,T}^{m-n})^T$$

$$= (x_1, x_2, \ldots, x_l, \overbrace{DT, DT, \ldots, DT}^{n-l}\ \overbrace{T,T,\ldots,T}^{m-n})^T.$$

Remark. Since T is temporary, they are gone. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & D^{-1} & 0 \\ 0 & 0 & D^{-1} \end{pmatrix}\begin{pmatrix} a \\ DT \\ DT \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D^{-1} & 0 \\ 0 & 0 & D^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}\begin{pmatrix} a \\ T \\ T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ T \\ T \end{pmatrix} = \begin{pmatrix} a \\ T \\ T \end{pmatrix} = a\ .$$

See also Eq. 1.1 and 1.2.

$$\therefore = (x_1, x_2, \dots, x_l, \overbrace{DT, DT, \dots, DT}^{n-l})^T$$
$$= diag(\overbrace{1,1,\dots,1}^{l}, \overbrace{D, D, \dots, D}^{n-l})(x_1, x_2, \dots, x_l, \overbrace{T, T, \dots, T}^{n-l})^T$$
$$= E_{ln}x \ .$$

a3) If $0 < m < l < n$ (projecting into higher dimensions),

$$x' = E_{lm}x = (E_{ml})^{-1}\ x = diag(\overbrace{1,1,\dots,1}^{m}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-m})(x_1, x_2, \dots, x_l)^T$$
$$= (x_1, x_2, \dots, x_l, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T.$$
$$\therefore\ E_{mn}x' = diag(\overbrace{1,1,\dots,1}^{m}, \overbrace{D, \mathrm{D}, \dots, D}^{m-n})(x_1, x_2, \dots, x_l, \overbrace{D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l}^{l-m}, \overbrace{T, T, \dots, T}^{n-l})^T$$
$$= (x_1, x_2, \dots, x_l, \overbrace{DT, DT, \dots, DT}^{n-l})^T$$
$$= diag(\overbrace{1,1,\dots,1}^{l}, \overbrace{D, D, \dots, D}^{n-l})(x_1, x_2, \dots, x_l, \overbrace{T, T, \dots, T}^{n-l})^T$$
$$= E_{ln}x \ .$$

b1) If $0 < n < m < l$ (projecting into lower dimensions),

$$x' = E_{lm}x = (E_{ml})^{-1}\ x = diag(\overbrace{1,1,\dots,1}^{m}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-m})(x_1, x_2, \dots, x_l)^T$$
$$= (x_1, x_2, \dots, x_l, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T.$$

$$\therefore\ E_{mn}x' = (E_{nm})^{-1}\ x'$$
$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{m-n}\ \overbrace{1,1,\dots,1}^{l-m})(x_1, x_2, \dots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T$$
$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{l-n})(x_1, x_2, \dots, x_l)^T$$
$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

b2) If $0 < n < l < m$ (projecting into lower dimensions),

$$x' = E_{lm}x = diag(\overbrace{1,1,\dots,1}^{l}, \overbrace{D,D,\dots,D}^{m-l})(x_1, x_2, \dots, x_l, \overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1, x_2, \dots, x_m, \overbrace{DT, DT, \dots, DT}^{m-l})^T.$$

$$\therefore\ E_{mn}x' = (E_{nm})^{-1}\ x'$$

$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{m-n})\ (x_1, x_2, \dots, x_m, \overbrace{D^{-1}T, D^{-1}T, \dots, D^{-1}T}^{m-l})^T$$

$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l, \overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{l-n})(x_1, x_2, \dots, x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

b3) If $0 < m < n < l$ (projecting into lower dimensions),

$$x' = E_{lm}x = (E_{ml})^{-1}\ x = diag(\overbrace{1,1,\dots,1}^{m}, \overbrace{D,\mathrm{D},\dots,D}^{l-m})(x_1, x_2, \dots, x_l)^T$$
$$= (x_1, x_2, \dots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T.$$

$$\therefore\ E_{mn}x' = diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D,\mathrm{D},\dots,D}^{n-m}\ \overbrace{1,1,\dots,1}^{n-l})(x_1, x_2, \dots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T$$
$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{l-n})(x_1, x_2, \dots, x_l)^T$$
$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

Proof of (ii), of associative law, is as follow: Since $(E_{kl}E_{lm})E_{mn} = E_{km}E_{mn} = E_{kn}$, and $E_{kl}(E_{lm}E_{mn}) = E_{kl}E_{ln} = E_{kn}$ from (i), $(E_{kl}E_{lm})E_{mn} = E_{kl}(E_{lm}E_{mn})$.

Proof of (iii) follows the rule of Eq. 2.6. Another proof for Eq. 3.4 is, from the formula (i), $f(AA^{-1}) = I = f(A)f(A^{-1})$. $\therefore f(A^{-1}) = f(A)^{-1}$, where $f(A^{-1}) = E_{ml}$ and $f(A)^{-1} = (E_{lm})^{-1}$.

Proof of (iv), it is trivial from (ii).

Proof of (v), $E_{lm}E_{mn}(E_{lm}E_{mn})^{-1} = E_{lm}E_{mn}(E_{mn})^{-1}(E_{lm})^{-1} = E_{ll} \equiv 1$.

At last, proof of (vi) is as follows. For $G = \{E_{lm}\}$, the scalar multiplication by 1 in field k holds as $s: 1\times G = \mathrm{G}\times 1 \rightarrow G$. It is compatible with the matrix multiplications in G. Then, $E_{mn} = I_m E_{mn} \equiv 1E_{mn} = E_{mn}1 \equiv E_{mn}I_n = E_{mn}$. Since it is 'mapping to itself' (in the narrower sense of the word of our discussion), it is equivalent to conventional unit matrices. □

Since this groupoid is homomorphism from (i), it can be considered as a representation of groupoid. Strictly speaking, it is automorphism. This will be proven later.

From another viewpoint

The groupoid mentioned above is partially defined, not for any two elements arbitrarily taken from G. We will not therefore consider it as a group. However, we have to take notice that group axioms do not claim that such a whole process (binary operation) should be done. To confirm it, let us try to give five conditions as group axioms as follows:

(1). We randomly take any two elements in a set G.

(2). For any two elements taken from G, the operation is closed in G, s. t. for any a, b, c in G, ab = c.

(3). For any a, b, c in G, $(ab)c = a(bc)$: associative law holds.

(4). There exists unique identity e.

(5). For each a in G, its inverse b exists s.t. $ab = ba = e$.

What we must pay attention to is whether the first condition should be included in axioms of groups. If accepting it, we should introduce a concept of axiom in probability theory. That is, in group theory, we assume the whole event for any two elements arbitrarily taken from G in the manner of probability theory, then define binary operation such for any elements taken from G at random. In other words, we should consider so-called sample space or measure theory for probability in group theory. Group axioms naturally do not claim such a process and another axiomatic system in probability theory.

Remark. In conventional algebra, the number of combinations is $N \times N$ (Descartes product) at most. However, the number of the operators' combinations is $N \times N \times N \times N$. Even in the case of $E_{lm}E_{mn}$, the number is $N \times N \times N$. So we must contemplate and rethink these issues.

For $f(AB)$,

$$\big(f(AB)\big)(x) = E_{lm}E_{mn}x$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l}\overbrace{1,1,\dots,1}^{n-m})diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,D,\dots,D}^{n-m})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,T}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

$$\therefore f(AB) = f(A)(B).$$

Chapter 4: The Groupoid Representation

Claim. *Equation of the groupoid is* $E_{lm}E_{mn} = E_{ln}$ *automorphism.*

Proof. Firtly, the automorphism is proved as follows. Let $f(A)$ be E_{lm}, $f(B)$ be E_{mn}.

a1) If $0 < l < m < n$ (projecting into higher dimensions),

For $f(A)(B)$,

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,T}^{m-l})^T.$$

Then $\big(f(B)\big)(x') = E_{mn}x' = diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,D,\dots,D}^{n-m})(x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l}\overbrace{T,T,\dots,T}^{n-m})^T$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,T}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

a2) If $0 < l < n < m$ (projecting into higher dimensions),

For $f(A)f(B)$,

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,T}^{m-l})^T$$

Then $\big(f(B)\big)(x') = E_{mn}x' = (E_{nm})^{-1}\, x'$

$$= diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D^{-1},D^{-1},\dots,D^{-1}}^{m-n})(x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l}\,\overbrace{T,T,\dots,T}^{m-n})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l}\,\overbrace{T,T,\dots,T}^{m-n})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

For $f(AB)$,

$$\big(f(AB)\big)(x) = (E_{lm}E_{mn})x$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{m-l})diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,D,\dots,D}^{m-n})(x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l}\overbrace{T,T,\dots,T}^{m-n})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l}\overbrace{T,T,\dots,T}^{m-n})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

$\therefore f(AB) = f(A)(B)$.

a3) If $0 < m < l < n$ (projecting into higher dimensions),

For $f(A)f(B)$,

$$(f(A))(x) = x' = E_{lm}x = (E_{ml})^{-1}\ x$$

$$= diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D^{-1},D^{-1},\dots,D^{-1}}^{l-m})(x_1,x_2,\dots,x_l)^T$$

$$= (x_1,x_2,\dots,x_l,D^{-1}x_{m+1},D^{-1}x_{m+2},\dots,D^{-1}x_l)^T.$$

Then, $\big(f(B)\big)(x') = E_{mn}x'$

$$=diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,\mathrm{D},\dots,D}^{m-n})(x_1,x_2,\dots,x_l,\overbrace{D^{-1}x_{m+1},D^{-1}x_{m+2},\dots,D^{-1}x_l}^{l-m},\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,D,\dots,D}^{n-l})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

For $f(AB)$,

$$\big(f(AB)\big)(x) = (E_{lm}E_{mn})x$$

$$= diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D^{-1},D^{-1},\dots,D^{-1}}^{l-m}\overbrace{1,1,\dots,1}^{n-l})diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,\mathrm{D},\dots,D}^{n-m})(x_1,x_2,\dots,x_l\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,\mathrm{D},\dots,D}^{n-l})(x_1,x_2,\dots,x_l\overbrace{T,T,\dots,T}^{n-l})^T$$

$$= E_{ln}x\ .$$

$$\therefore\ f(AB) = f(A)(B).$$

b1) If $0 < n < m < l$ (projecting into lower dimensions),

For $f(A)f(B)$,

$$(f(A))(x) = x'z = E_{lm}x = (E_{ml})^{-1}\, x = diag(\overbrace{1,1,\dots,1}^{m}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-m})(x_1, x_2, \dots, x_l)^T$$

$$= (x_1, x_2, \dots, x_l, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T.$$

Then,

$$\big(f(B)\big)(x) = E_{mn}x' = (E_{nm})^{-1}\, x'$$

$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{m-n}\ \overbrace{1,1,\dots,1}^{l-m})(x_1, x_2, \dots, x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, \dots, D^{-1}x_l)^T$$

$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{l-n})(x_1, x_2, \dots, x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

For $f(AB)$,

$$\big(f(AB)\big)(x) = (E_{lm}E_{mn})x$$

$$= diag(\overbrace{1,1,\dots,1}^{m}, \overbrace{D^{-1}, D^{-1}, \dots, D^{-1}}^{l-m})(x_1, x_2, \dots, x_l)^T \; diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{m-n}\overbrace{1,1,\dots,1}^{l-m})$$

$$= (x_1, x_2, \dots, x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, \dots, D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\dots,1}^{n}, \overbrace{D^{-1}D^{-1}, \dots, D^{-1}}^{l-n})(x_1, x_2, \dots, x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x \; .$$

$\therefore f(AB) = f(A)(B).$

b2) If $0 < n < l < m$ (projecting into lower dimensions),

For $f(A)f(B)$,

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1,1,\dots,1}^{m},\overbrace{D,\mathrm{D},\dots,D}^{l-m})(x_1,x_2,\dots,x_l,\overbrace{T,T,\dots,T}^{l-m})^T$$

$$= (x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l})^T.$$

Then,

$$(f(B))(x') = E_{mn}x' = (E_{nm})^{-1}\ x'$$

$$= diag(\overbrace{1,1,\dots,1}^{n},\overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{m-n})(x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\dots,D^{-1}x_l,\overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\dots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\dots,1}^{n},\overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{l-n})(x_1,x_2,\dots,x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

For $f(AB)$,

$$\big(f(AB)\big)(x) = (E_{lm}E_{mn})x$$

$$= diag(\overbrace{1,1,\dots,1}^{l},\overbrace{D,\mathrm{D},\dots,D}^{m-l})diag(\overbrace{1,1,\dots,1}^{n},\overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{m-n})(x_1,x_2,\dots,x_l,\overbrace{DT,DT,\dots,DT}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\dots,D^{-1}x_l,\overbrace{T,T,\dots,T}^{m-l})^T$$

$$= (x_1,x_2,\dots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\dots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\dots,1}^{n},\overbrace{D^{-1}D^{-1},\dots,D^{-1}}^{l-n})(x_1,x_2,\dots,x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

$$\therefore f(AB) = f(A)(B).$$

b3) If $0 < m < n < l$ (projecting into lower dimensions),

For $f(A)f(B)$,

$$(f(A))(x) = x' = E_{lm}x = (E_{ml})^{-1}\ x$$

$$= diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,\mathrm{D},\ldots,D}^{l-m})(x_1,x_2,\ldots,x_l)^T$$

$$= (x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T.$$

Then,

$$\big(f(B)\big)(x') = E_{mn}x'$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D,\mathrm{D},\ldots,D}^{n-m}\ \overbrace{1,1,\ldots,1}^{n-l})(x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1}D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

For $f(AB)$,

$$(f(AB))(x) = (E_{lm}E_{mn})x$$

$$= diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D,\mathrm{D},\ldots,D}^{l-m})diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D,\mathrm{D},\ldots,D}^{n-m}\overbrace{1,1,\ldots,1}^{n-l})(x_1,x_2,\ldots,x_l)^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n}, \overbrace{D^{-1}D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= (E_{nl})^{-1}x = E_{ln}x\ .$$

$$\therefore f(AB) = f(A)(B).$$

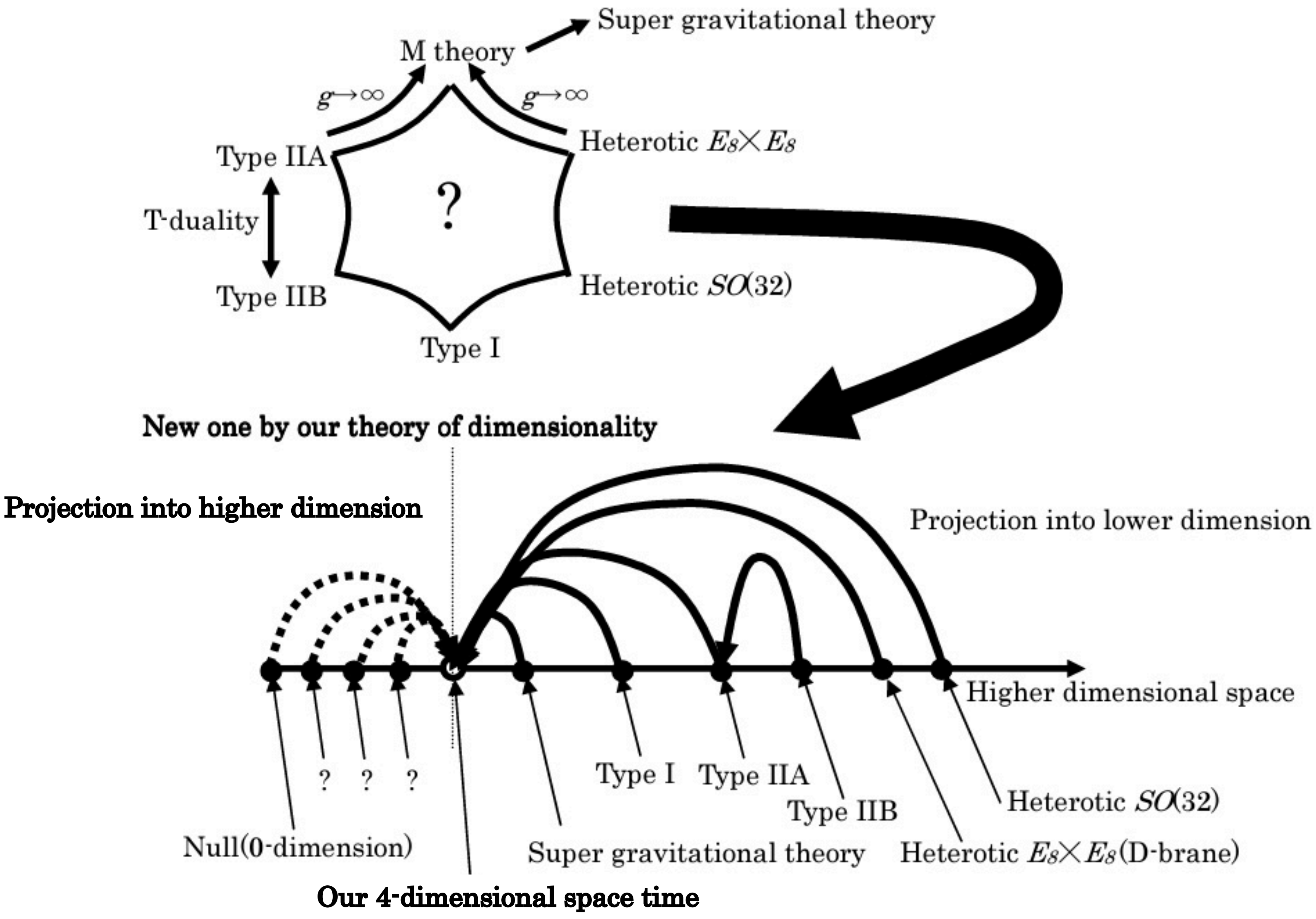
Epilogue: The Invariant and Symmetry — Towards Noether’s Theorem

- **Invariant:** In theoretical physics, an invariant means a physical system unchanged under mathematical operation. It is called also symmetry.
- **Noether’s Theorem:** It states that every differential symmetry of the action of a physical system with conservative forces has a corresponding conservation law.
- Noether’s theorem actually holds not only differential symmetries but also discrete symmetries. Parity and selection rule in quantum theory are those examples.

What is the invariant in the groupoid that we have discussed?

- The invariant is conservation of the degree of freedom: wherever a point is projected, its degree of freedom is conserved.
- It suggests that if higher dimensional physics were described by the groupoid, we might find an unknown physical conservation law.

What is a preferable unification of string theories?



- The upper model is by M-Theory.
- The lower one is by so-to-speak our theory of dimensionality.

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- For a quick grasp of this research, see also:
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- http://dx.doi.org/10.13140/RG.2.2.21097.93285