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What Comes Beyond the Standard Models?

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The gauge coupling unification in Grand Unified Theories based on the group E_8

based on K.S., ArXiv:2305.01295 [hep-ph]

MSSM is the simplest supersymmetric extension of the Standard Model. It is a gauge theory with the group $SU_3 \times SU_2 \times U_1$ and softly broken supersymmetry. Consequently, there are 3 gauge coupling constants e_3, e_2, and e_1 in MSSM (their number is equal to the number of factors in the gauge group). Quarks, leptons, and Higgs fields are components of the chiral matter superfields:

Superfield	SU_3	SU_2	U_1 (Y)	Superfield	SU_3	SU_2	U_1 (Y)
$3\times Q$	$\bar{3}$	2	$-1/6$	$3\times N$	1	1	0
$3\times U$	3	1	$2/3$	$3\times E$	1	1	-1
$3\times D$	3	1	$-1/3$	H_d	1	2	$1/2$
$3\times L$	1	2	$1/2$	H_u	1	2	$-1/2$

where for the superfields which include left quarks and leptons we use the brief notations

$$Q = \begin{pmatrix} \widetilde{U} \\ \widetilde{D} \end{pmatrix}; \qquad L = \begin{pmatrix} \widetilde{N} \\ \widetilde{E} \end{pmatrix}.$$

Quantum numbers of various MSSM superfields are not accidental. They satisfy, e.g., the anomaly cancellation conditions

$$\mathrm{tr}\Big(T^A\{T^B,T^C\}\Big)=0,$$

where T^A are the generators of the representation in which the chiral matter superfields lie. For MSSM the nontrivial relations needed for this equation to be satisfied have the form

$$2\times SU_3+U_1:\quad \frac{1}{4}\mathrm{tr}(\lambda^A\lambda^B)\Big(Y_U+Y_D+2\cdot Y_Q\Big)=\frac{1}{4}\mathrm{tr}(\lambda^A\lambda^B)\Big(\frac{2}{3}-\frac{1}{3}-2\cdot\frac{1}{6}\Big)=0;$$

$$2\times SU_2+U_1:\quad \frac{1}{4}(\sigma^\alpha\sigma^\beta)\Big(3\cdot Y_Q+Y_L\Big)=\frac{1}{4}(\sigma^\alpha\sigma^\beta)\Big(-3\cdot\frac{1}{6}+\frac{1}{2}\Big)=0;$$

$$3\times U_1:\quad \sum Y^3=3\cdot Y_U^3+3\cdot Y_D^3+Y_E^3+6\cdot Y_Q^3+2\cdot Y_L^3$$
$$=3\cdot\Big(\frac{2}{3}\Big)^3+3\cdot\Big(-\frac{1}{3}\Big)^3+(-1)^3+6\cdot\Big(-\frac{1}{6}\Big)^3+2\cdot\Big(\frac{1}{2}\Big)^3=0.$$

(The contributions of H_d and H_u cancel each other and are not written here.) Therefore, there is a question if these quantum numbers are accidental and how they appear.

Similarly to the nonsupersymmetric case first considered in the paper

H.Georgi and S.L.Glashow, Phys. Rev. Lett. **32** (1974), 438.

the analysis of MSSM superfield quantum numbers demonstrates that the superfields of a single generation (including the right neutrino) can be accomodated in 3 irreducible representations of the group SU_5

$$1 + 5 + \overline{10}$$

$$1 \sim N; \quad 5_i \sim \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \widetilde{E} \\ -\widetilde{N} \end{pmatrix}; \quad \overline{10}^{ij} \sim \begin{pmatrix} 0 & U_3 & -U_2 & \widetilde{U}^1 & \widetilde{D}^1 \\ -U_3 & 0 & U_1 & \widetilde{U}^2 & \widetilde{D}^2 \\ U_2 & -U_1 & 0 & \widetilde{U}^3 & \widetilde{D}^3 \\ -\widetilde{U}^1 & -\widetilde{U}^2 & -\widetilde{U}_3 & 0 & E \\ -\widetilde{D}^1 & -\widetilde{D}^2 & -\widetilde{D}^3 & -E & 0 \end{pmatrix}.$$

In this case after the symmetry breaking $SU_5 \to SU_3 \times SU_2 \times U_1$ all fields of the low-energy theory will have correct quantum numbers.

By a vacuum expectation value of the Higgs field in the adjoint representation 24 the SU_5 symmetry can be broken down to the subgroup $SU_3\times SU_2\times U_1$ with the elements

$$\omega_5 = \begin{pmatrix} \omega_3 e^{-i\beta_Y/3} & 0\\ 0 & \omega_2^* e^{i\beta_Y/2} \end{pmatrix} \in SU_3\times SU_2\times U_1 \subset SU_5.$$

Then, from the SU_5 tensor transformations

$$N\to N;\qquad 5_i \to (\omega_5)_i{}^j 5_j;\qquad \overline{10}^{ij}\to (\omega_5^*)^k{}_i(\omega_5^*)^l{}_j\,\overline{10}^{kl}$$

one obtains that with respect to the subgroup $SU_3\times SU_2\times U_1$ all chiral superfields have the same quantum numbers as the MSSM superfields.

The further symmetry breaking $SU_3\times SU_2\times U_1 \to SU_3\times U_1^{em}$ is usually made by vacuum expectation values of the Higgs superfields in the representations 5 and $\bar 5$. However, in this case the doublet-triplet splitting requires fine tuning.

The anomaly cancellation in this model occurs due to the relation

$$\mbox{tr}\Big(T^A\{T^B,T^C\}\Big)\Big|_5 + \mbox{tr}\Big(T^A\{T^B,T^C\}\Big)\Big|_{\overline{10}} = 0.$$

Because the group SU_5 is simple, there is the only gauge coupling constant e_5 in the SU_5 Grand Unified Theory (GUT). This implies that in the low-energy $SU_3 \times SU_3 \times U_1$ theory 3 coupling constants should be related to each other. This relation is written as

$$\alpha_2 = \alpha_3; \qquad \sin^2\theta_W = 3/8,$$

where $\mathrm{tg}\,\theta_W \equiv e_1/e_2$. If we introduce the notation $\alpha_1 \equiv \dfrac{5}{3}\cdot\dfrac{e_1^2}{4\pi}$, then the gauge coupling unification condition takes the simplest form $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5$. This condition is in a good agreement with the well-known renormalization group behaviour of the running gauge couplings in MSSM.

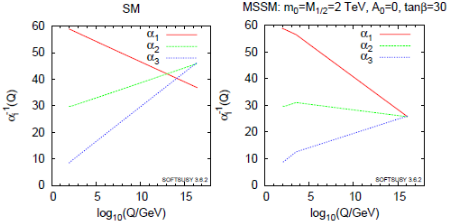


Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

The field content of the SU_5 GUT implies a possibility of the existence of a theory with a wider SO_{10} symmetry

H.Fritzsch, P.Minkowski, Annals Phys. **93** (1975), 193;
H.Georgi, AIP Conf. Proc. **23** (1975), 575.

because the superfields of a single generation can be accommodated into a single irreducible (spinor) representation

$$\overline{16}\Big|_{SO_{10}} = 1(5) + 5(-3) + \overline{10}(1)\Big|_{SU_5\times U_1},$$

However, the symmetry breaking pattern

$$SO_{10} \to SU_5 \to SU_3 \times SU_2 \times U_1$$

has some drawbacks. In particular, for the symmetry breaking one needs (super)fields in sufficiently large representations (no less than $45\Big|_{SO_{10}}$ and $24\Big|_{SU_5}$), and the simplest (supersymmetric) SU_5 model is excluded by the modern experimental limits on the proton lifetime.

A more convenient symmetry breaking pattern is

$$SO_{10} \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1.$$

It corresponds to the flipped SU_5 model

S.M.Barr, Phys. Lett. B **112** (1982), 219; I.Antoniadis, J.R.Ellis, J.S.Hagelin, D.V.Nanopoulos, Phys. Lett. B **194** (1987), 231; B.A.Campbell, J.R.Ellis, J.S.Hagelin, D.V.Nanopoulos, K.A.Olive, Phys. Lett. B **197** (1987), 355; J.R.Ellis, J.S.Hagelin, S.Kelley, D.V.Nanopoulos, Nucl. Phys. B **311** (1988), 1.

In this case the chiral matter superfields are accommodated in the representations $3\times\Big(\overline{10}(1)+5(-3)+1(5)\Big)$ in a different way,

$$1\sim E;\qquad 5_i\sim\begin{pmatrix} U_1\\ U_2\\ U_3\\ \widetilde{E}\\ -\widetilde{N}\end{pmatrix};\qquad \overline{10}^{ij}\sim\begin{pmatrix} 0 & D_3 & -D_2 & \widetilde{U}^1 & \widetilde{D}^1\\ -D_3 & 0 & D_1 & \widetilde{U}^2 & \widetilde{D}^2\\ D_2 & -D_1 & 0 & \widetilde{U}^3 & \widetilde{D}^3\\ -\widetilde{U}^1 & -\widetilde{U}^2 & -\widetilde{U}^3 & 0 & N\\ -\widetilde{D}^1 & -\widetilde{D}^2 & -\widetilde{D}^3 & -N & 0\end{pmatrix}.$$

We see that the superfields corresponding to the right up and down quarks and leptons are swapped. (That is why this model is called “flipped”.)

The flipped SU_5 model

The $SU_5 \times U_1$ symmetry is broken down to $SU_3 \times SU_2 \times U_1^{(Y)}$ by vacuum expectation values of Higgses in the representations $10(-1)$ and $\overline{10}(1)$, and the group $U_1^{(Y)}$ appears as a superposition of the $SU(5)$ transformations with

$$\omega_5 = \exp\left\{ \frac{i\alpha_Y}{30} \begin{pmatrix} 2\cdot 1_3 & 0 \\ 0 & -3\cdot 1_2 \end{pmatrix} \right\}$$

and the U_1 transformations with $\omega_1 = \exp(-iQ\alpha_Y/5)$, where Q is the U_1 charge normalized as was pointed above. This model

1. allows to naturally split Higgs douplet and triplet;
2. does not require higher representations for the breaking of the SU_5 symmetry;
3. satisfies present limits on the proton lifetime.

The flipped SU_5 model has 2 coupling constants e_5 and e_1. However, if it is considered as a remnant of the SO_{10} theory, then they are related to each other as

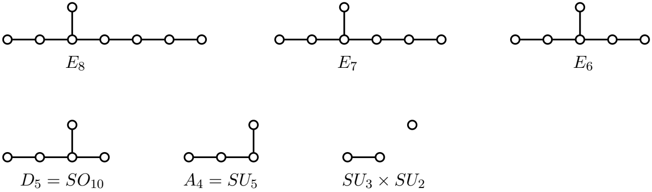
$$e_5 = \frac{e_{10}}{\sqrt{2}}; \qquad e_1 = \frac{e_5}{2\sqrt{10}} = \frac{e_{10}}{4\sqrt{5}}.$$

Then for the residual $SU_3 \times SU_2 \times U_1$ theory we obtain the standard relations

$$\alpha_2 = \alpha_3; \qquad \sin^2\theta_W = 3/8.$$

The E-series of the simple compact Lie algebras and the flipped E_8 GUT

It is known that the Lie algebras used for constructing various GUTs belong to the E-series if we also include in it some classical Lie algebras



We will investigate a possibility of constructing GUT based on the group E_8 assuming that

1. The symmetry breaking pattern is

$$E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \times U_1 \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1.$$

2. Vacuum expectation values responsible for the various symmetry breakings can be acquired only by certain parts of the fundamental representation of the group E_8 (of the dimension 248).

Let us construct the Γ-matrices in the space of a dimension D with the Euclidean signature, which satify the condition

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \cdot 1,$$

For an even D they have the size $2^{D/2} \times 2^{D/2}$, and for an odd D their size is $2^{(D-1)/2} \times 2^{(D-1)/2}$. For $D = 2, 3$ as the Γ-matrixes one can choose the Pauli matrices,

$$\Gamma_1^{(2)} = \sigma_1; \quad \Gamma_2^{(2)} = \sigma_2; \qquad \qquad \Gamma_1^{(3)} = \sigma_1; \quad \Gamma_2^{(3)} = \sigma_2; \quad \Gamma_3^{(3)} = \sigma_3.$$

For the other values of D the Γ-matrices are constructed with the help of mathematical induction. Suppose that they have been constructed in an odd dimension D. Then in the even dimension $D+1$ the Γ-matrices are defined as

$$\Gamma_i^{(D+1)} = \begin{pmatrix} 0 & \Gamma_i^{(D)} \\ \Gamma_i^{(D)} & 0 \end{pmatrix}, \quad i = 1, \ldots, D; \qquad \Gamma_{D+1}^{(D+1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and have the size in two times larger, than in the previous (odd) dimension. Moreover, in this case there is the matrix

$$\Gamma_{D+2}^{(D+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix $\Gamma^{(D+1)}_{D+2}$ satisfies the conditions

$$\{\Gamma^{(D+1)}_{D+2}, \Gamma^{(D+1)}_{i}\} = 0, \quad i = 1, \ldots D+1; \qquad \big(\Gamma^{(D+1)}_{D+2}\big)^2 = 1.$$

Therefore, in the next (odd) dimension $D+2$, as Γ-matrices, one can take the Γ-matrices from the previous (even) dimension, $\Gamma^{(D+2)}_{i} \equiv \Gamma^{(D+1)}_{i}$ (where $i = 1, \ldots D+1$), and add to them $\Gamma^{(D+2)}_{D+2} \equiv \Gamma^{(D+1)}_{D+2}$. This completes the induction step.

The Γ-matrices constructed in this way are Hermitian, $(\Gamma_i)^+ = \Gamma_i$. For even i they are antisymmetric, and for odd i they are symmetric. In an even dimension D the charge conjugation matrix is defined as

$$C \equiv \Gamma_1 \Gamma_3 \ldots \Gamma_{D-1}$$

and satisfies the conditions

$$C\,\Gamma_i\,C^{-1} = -(-1)^{D/2}\,(\Gamma_i)^T; \qquad C^{-1} = C^+ = C^T = (-1)^{D(D-2)/8}\,C.$$

Also (for even D) the following equations take place:

$$(\Gamma_{i_1 i_2 \ldots i_k} C)^T = (-1)^{(D-2k)(D-2k-2)/8}\,\Gamma_{i_1 i_2 \ldots i_k} C;$$

$$(\Gamma_{i_1 i_2 \ldots i_k} \Gamma_{D+1} C)^T = (-1)^{(D-2k)(D-2k+2)/8}\,\Gamma_{i_1 i_2 \ldots i_k} \Gamma_{D+1} C.$$

Notations

The generators of the fundamental representation are denoted by t_A, where $A = 1, \ldots, \dim G$. They are normalized by the condition

$$\mathrm{tr}(t_A t_B) = \frac{1}{2} g_{AB},$$

where $g_{AB} = g_{BA}$ is a metric. The matrix inverse to g_{AB} is denoted by g^{AB}. The generators T_A of an arbitrary representation R satisfy the equations

$$\mathrm{tr}(T_A T_B) = T(R) g_{AB}; \qquad g^{AB}(T_A T_B)_i{}^j = C(R)_i{}^j; \qquad [T_A, T_B] = i f_{AB}{}^C T_C,$$

where $f_{AB}{}^C$ are the structure constants. The expression $f_{ABC} \equiv g_{CD} f_{AB}{}^D$ is totally antisymmetric, and

$$(T_{Adj\,A})^C{}_B = i f_{AB}{}^C; \qquad C_2\, g_{AB} \equiv -f_{AC}{}^D f_{BD}{}^C.$$

In particular, from these equations we obtain $g^{AB}[T_A, [T_B, T_C]] = C_2 T_C$.
For irreducible representations

$$C(R)_i{}^j = C(R)\, \delta_i{}^j, \quad \text{where} \quad C(R) = T(R) \cdot \frac{\dim G}{\dim R}.$$

The fundamental representation of the group E_8 coincides with the adjoint representation and has the dimension 248. The group E_8 has the maximal subgroup $SO_{16} \subset E_8$, with respect to which

$$248\Big|_{E_8} = 120 + 128\Big|_{SO_{16}}.$$

Here 120 is the adjoint representation of SO_{16}, and 128 is its representation by Majorana-Weyl (right, for the definiteness) spinors. Therefore,

$$t_{\boldsymbol{A}} = \{t_a,\, t_{ij}\},$$

where $i, j = 1, \ldots, 16$ and $a = 1, \ldots, 128$. The commutation relations of the group E_8 are written as

$$E_8 \quad \begin{cases} [t_{ij}, t_{kl}] = \dfrac{i}{\sqrt{120}}\Big(\delta_{il} t_{jk} - \delta_{jl} t_{ik} - \delta_{ik} t_{jl} + \delta_{jk} t_{il}\Big); \\ [t_{ij}, t_a] = -\dfrac{i}{\sqrt{480}}(\Gamma_{ij}^{(16)})_a{}^b t_b; \\ [t_a, t_b] = -\dfrac{i}{2\sqrt{480}}(\Gamma_{ij}^{(16)} C^{(16)})_{ab} t_{ij}. \end{cases}$$

Here $C^{(16)}$ is the (symmetric) charge conjugation matrix in $D = 16$, and the matrices $\Gamma^{(16)}_{ij}C^{(16)}$ are antisymmetric. The corresponding metric has the form

$$g_{\boldsymbol{AB}} \to \begin{pmatrix} \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} & 0 \\ 0 & (C^{(16)})_{ab} \end{pmatrix};$$

$$g^{\boldsymbol{AB}} \to \begin{pmatrix} \frac{1}{4}\big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\big) & 0 \\ 0 & (C^{(16)})^{ab} \end{pmatrix}.$$

In particular, it is easy to verify the identity

$$g^{\boldsymbol{AB}} t_{\boldsymbol{A}} t_{\boldsymbol{B}} = \frac{1}{2} t_{ij} t_{ij} + (C^{(16)})^{ab} t_a t_b = \frac{1}{2},$$

which is certainly equivalent to the equations

$$\frac{1}{2}[t_{ij}, [t_{ij}, t_{kl}]] + (C^{(16)})^{ab}[t_a, [t_b, t_{kl}]] = \frac{1}{2} t_{kl};$$
$$\frac{1}{2}[t_{ij}, [t_{ij}, t_d]] + (C^{(16)})^{ab}[t_a, [t_b, t_d]] = \frac{1}{2} t_d.$$

To describe the group E_7, we use the subgroup $SO_{12}\times SO_3\subset E_7$. Then

$$56\Big|_{E_7} = [12,2]+[32,1]\Big|_{SO_{12}\times SO_3};$$
$$133\Big|_{E_7} = [1,3]+[32',2]+[66,1]\Big|_{SO_{12}\times SO_3},$$

where 32 and $32'$ are right and left spinor representations of SO_{12}. The indices of the left SO_{12} spinors are denoted by dots. Therefore,

$$t_{\boldsymbol{A}} = \Big\{t_{ij}, t_\alpha, t_{a\dot A}\Big\},\quad \mbox{where}\quad a,b=1,2;\quad \alpha,\beta=1,\ldots,3;\quad i,j=1,\ldots,12.$$

$$E_7\quad \left\{\begin{array}{l} [t_\alpha,t_\beta] = \dfrac{i}{\sqrt{12}}\varepsilon_{\alpha\beta\gamma}t_\gamma;\qquad [t_\alpha,t_{ij}]=0;\\[2mm] [t_\alpha,t_{a\dot A}] = -\dfrac{1}{2\sqrt{12}}(\sigma_\alpha)_a{}^b t_{b\dot A};\qquad [t_{ij},t_{a\dot A}] = -\dfrac{i}{2\sqrt{24}}(\Gamma^{(12)}_{ij})_{\dot A}{}^{\dot B}t_{a\dot B};\\[2mm] [t_{ij},t_{kl}] = \dfrac{i}{\sqrt{24}}\Big(\delta_{il}t_{jk}-\delta_{jl}t_{ik}-\delta_{ik}t_{jl}+\delta_{jk}t_{il}\Big);\\[2mm] [t_{a\dot A},t_{b\dot B}] = \dfrac{i}{4\sqrt{24}}(\sigma_2)_{ab}(\Gamma^{(12)}_{ij}C^{(12)})_{\dot A\dot B}t_{ij} + \dfrac{1}{2\sqrt{12}}(C^{(12)})_{\dot A\dot B}(\sigma_\alpha\sigma_2)_{ab}t_\alpha. \end{array}\right.$$

The corresponding metric has the form

$$g_{\boldsymbol{AB}} \to \begin{pmatrix} \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} & 0 & 0 \\ 0 & \delta_{\alpha\beta} & 0 \\ 0 & 0 & -(\sigma_2)_{ab}(C^{(12)})_{\dot{A}\dot{B}} \end{pmatrix};$$

$$g^{\boldsymbol{AB}} \to \begin{pmatrix} \frac{1}{4}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big) & 0 & 0 \\ 0 & \delta_{\alpha\beta} & 0 \\ 0 & 0 & (\sigma_2)^{ab}(C^{(12)})^{\dot{A}\dot{B}} \end{pmatrix}.$$

Note that the matrices $\sigma_2 = i\sigma_1\sigma_3$ and $C^{(12)}$ are antisymmetric, while the matrices $\sigma_\alpha\sigma_2$ and $\Gamma_{ij}^{(12)}C^{(12)}$ are symmetric,

$$(C^{(12)})^T = -C^{(12)}; \qquad (C^{(12)})^2 = -1; \qquad (\Gamma_{ij}^{(12)}C^{(12)})^T = \Gamma_{ij}^{(12)}C^{(12)}.$$

Therefore, the above metric is really symmetric.

The group E_7

In the explicit form the generators of the fundamental representation 56 are written as

$$t_{ij} = \frac{i}{\sqrt{24}} \begin{pmatrix} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\delta_a^b & 0 \\ 0 & \frac{1}{2}(\Gamma_{ij}^{(12)})_A{}^B \end{pmatrix};$$

$$t_\alpha = \frac{1}{2\sqrt{12}} \begin{pmatrix} \delta_{kl}(\sigma_\alpha)_a{}^b & 0 \\ 0 & 0 \end{pmatrix};$$

$$t_{d\dot{D}} = \frac{i}{2\sqrt{12}} \begin{pmatrix} 0 & (\sigma_2)_{da}(\Gamma_k^{(12)})_{\dot{D}}{}^B \\ (\Gamma_l^{(12)} C^{(12)})_{A\dot{D}}\delta_d^b & 0 \end{pmatrix}.$$

As a check, one can verify that

$$C(56) \equiv g^{\boldsymbol{AB}} t_{\boldsymbol{A}} t_{\boldsymbol{B}} = \frac{1}{2} t_{ij} t_{ij} + t_\alpha t_\alpha + (\sigma_2)^{ab} (C^{(12)})^{\dot{A}\dot{B}} t_{a\dot{A}} t_{b\dot{B}} = \frac{19}{16} = \frac{1}{2} \cdot \frac{133}{56}.$$

Let us investigate if it is possible to break the E_8 symmetry by the vacuum expectation value of the representation 248. For this purpose we consider the embedding

$$E_8 \supset SO_{16} \supset \underbrace{SO_{12} \times SO_3}_{\subset E_7} \times SO_3,$$

for which

$$248\Big|_{E_8} = 120 + 128\Big|_{SO_{16}}$$
$$= \underbrace{[1,1,3]}_{[1,3]} + \underbrace{[1,3,1] + [66,1,1] + [32',2,1]}_{+[133,1]} + \underbrace{[12,2,2] + [32,1,2]}_{+[56,2]\Big|_{E_7\times SO_3}}\Big|_{SO_{12}\times SO_3\times SO_3}.$$

Let us present a scalar field in the representation 248 in the form

$$\varphi = \varphi_{\boldsymbol{A}}\, g^{\boldsymbol{AB}} t_{\boldsymbol{B}} = \frac{1}{2}\varphi_{ij} t_{ij} + \varphi_a (C^{(16)})^{ab} t_b$$

and suppose that

$$\left(\varphi_{13,14}\right)_0 = \left(\varphi_{15,16}\right)_0 = v_8.$$

Let us construct the corresponding little group. By definition, a vacuum expectation value should be invariant under its transformations. Therefore, it is necessary to find all E_8 generators which commute with $\varphi_0 = v_8\big(t_{13,14} + t_{15,16}\big)$. Evidently, t_{ij} commute with φ_0 if $i,j = 1,\ldots,12$. These generators form the subgroup SO_{12}. Also the little group includes

$$\widetilde{t}_1 \equiv \frac{1}{\sqrt{2}}\big(t_{13,16} - t_{14,15}\big); \qquad \widetilde{t}_2 \equiv \frac{1}{\sqrt{2}}\big(- t_{13,15} - t_{14,16}\big);$$
$$\widetilde{t}_3 \equiv \frac{1}{\sqrt{2}}\big(t_{13,14} - t_{15,16}\big); \qquad \widetilde{t}_3' \equiv \frac{1}{\sqrt{2}}\big(- t_{13,14} - t_{15,16}\big).$$

They form the subgroup $SO_3 \times U_1$ of the little group,

$$[\widetilde{t}_3', \widetilde{t}_\alpha] = 0; \qquad [\widetilde{t}_\alpha, \widetilde{t}_\beta] = \frac{i}{\sqrt{60}}\varepsilon_{\alpha\beta\gamma}\widetilde{t}_\gamma.$$

However, the little group is wider than $SO_{12}\times SO_3 \times U_1$ because some generators t_a also commute with the vacuum expectation value. Really,

$$[\varphi_0, t_a] = v_8\, \big[t_{13,14} + t_{15,16}, t_a\big] = -\frac{i v_8}{2\sqrt{120}}\Big(\Gamma^{(16)}_{13,14} + \Gamma^{(16)}_{15,16}\Big)_a{}^b\, t_b.$$

Substituting the explicit form of the Γ-matrices we see that

$$-\frac{i}{2}\Big(\Gamma^{(16)}_{13,14} + \Gamma^{(16)}_{15,16}\Big) = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \cdot \frac{1}{2}\Big(1 + \Gamma^{(12)}_{13}\Big).$$

This implies that the generators t_a which belong to the little group form two left SO_{12} spinors (each of them having 32 nontrivial components). It is possible to verify that with respect to SO_3 they lie in the spinor representation 2. Therefore, we obtain the little group $E_7 \times U_1$ because

$$133\Big|_{E_7} = [1,3] + [32',2] + [66,1]\Big|_{SO_{12}\times SO_3}$$

Thus,

$$E_8 \to E_7 \times U_1.$$

Now, let us relate 2 coupling constants of the resulting theory with the original coupling constant e_8. Comparing the commutation relations of the generators t_{ij} for the groups E_8 and E_7 we see that

$$t_{ij}\Big|_{E_8} = \frac{1}{\sqrt{5}} t_{ij}\Big|_{E_7}.$$

Because $A_\mu = ieA_\mu^{\boldsymbol{A}} t_{\boldsymbol{A}}$ and the generators t_{ij} are normalized by the same condition

$$\mathrm{tr}(t_{ij}t_{kl}) = \frac{1}{2}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big),$$

the coupling constants for the groups E_7 and E_8 are related as

$$e_7 = \frac{e_8}{\sqrt{5}}.$$

Therefore, we should only find the value of the coupling constant $e_1^{(7)}$ which corresponds to the subgroup U_1. This constant depends on the normalization of the U_1 charge. Let us choose the SO_3 generators in the subgroup $E_7 \times SO_3 \subset E_8$ in such a way that

$$t'_\alpha\Big|_{R=2} = \sigma_\alpha; \qquad \Big(t'_\alpha\Big)_{\beta\gamma}\Big|_{R=3} = -2i\varepsilon_{\alpha\beta\gamma}.$$

In this case

$$[t'_\alpha, t'_\beta] = 2i\varepsilon_{\alpha\beta\gamma} t'_\gamma.$$

As a generator of the U_1 component of the little group we take t'_3. Then

$$248\Big|_{E_8} = 1(0) + 1(2) + 1(-2) + 133(0) + 56(1) + 56(-1)\Big|_{E_7\times U_1}.$$

The generators of the SO_3 subgroup in $E_7 \times SO_3 \subset E_8$ normalized in the same way as all E_8 generators satisfy the commutation relations

$$[\widetilde{t}'_\alpha, \widetilde{t}'_\beta] = \frac{i}{\sqrt{60}} \varepsilon_{\alpha\beta\gamma} \widetilde{t}'_\gamma.$$

Comparing them with the commutation relations for t'_α at the previous slide we see that

$$\widetilde{t}'_3 = \frac{1}{4\sqrt{15}} t'_3.$$

Therefore, the corresponding couplings are related by the equation

$$e_1^{(7)} = \frac{e_8}{4\sqrt{15}} = \frac{e_7}{4\sqrt{3}}.$$

Thus, at this stage of the symmetry breaking we finally obtain

$$\alpha_7 = \frac{\alpha_8}{5}; \qquad \alpha_1^{(7)} = \frac{\alpha_7}{48}.$$

For investigating the next symmetry breaking stage $E_7 \times U_1 \to E_6 \times U_1$ we will need some information about the group E_6, which is presented below.

The group E_6 has the maximal subgroup $SO_{10}\times U_1$, with respect to which

$$27\Big|_{E_6} = 1(4) + 10(-2) + 16(1)\Big|_{SO_{10}\times U_1};$$
$$\overline{27}\Big|_{E_6} = 1(-4) + 10(2) + \overline{16}(-1)\Big|_{SO_{10}\times U_1};$$
$$78\Big|_{E_6} = 1(0) + 16(-3) + \overline{16}(3) + 45(0)\Big|_{SO_{10}\times U_1},$$

where 16 and $\overline{16}$ are the right and left spinor representations of SO_{10}. However, now we will use a single spinor index $a = 1,\ldots,32$, so that

$$t_{\boldsymbol{A}} = \{t_{ij}, t_a, t\},$$

$$E_6\quad \begin{cases} [t_{ij}, t_{kl}] = \dfrac{i}{\sqrt{12}}\Big(\delta_{il}t_{jk} - \delta_{jl}t_{ik} - \delta_{ik}t_{jl} + \delta_{jk}t_{il}\Big); \\[2ex] [t_{ij}, t] = 0; \qquad [t, t_a] = \dfrac{1}{4}(\Gamma^{(10)}_{11})_a{}^b t_b; \qquad [t_{ij}, t_a] = -\dfrac{i}{2\sqrt{12}}(\Gamma^{(10)}_{ij})_a{}^b t_b; \\[2ex] [t_a, t_b] = -\dfrac{i}{4\sqrt{12}}(\Gamma^{(10)}_{ij}C^{(10)})_{ab}t_{ij} + \dfrac{1}{4}(\Gamma^{(10)}_{11}C^{(10)})_{ab}t. \end{cases}$$

In this case the metric has the form

$$g_{AB} \to \begin{pmatrix} \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} & 0 & 0 \\ 0 & (C^{(10)})_{ab} & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$g^{AB} \to \begin{pmatrix} \frac{1}{4}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big) & 0 & 0 \\ 0 & (C^{(10)})^{ab} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and in $D = 10$

$$(C^{(10)})^T = C^{(10)}; \qquad (C^{10})^2 = 1;$$

$$(\Gamma^{(10)}_{ij} C^{(10)})^T = -\Gamma^{(10)}_{ij} C^{(10)}; \qquad (\Gamma^{(10)}_{11} C^{(10)})^T = -\Gamma^{(10)}_{11} C^{(10)}.$$

The generators of the fundamental representation 27 have the form

$$t_{ij} = \frac{i}{\sqrt{12}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} & 0 \\ 0 & 0 & \frac{1}{4}\Big[\Gamma^{(10)}_{ij}(1+\Gamma^{(10)}_{11})\Big]_a{}^b \end{pmatrix};$$

$$t = \frac{1}{12} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2\delta_{kl} & 0 \\ 0 & 0 & \frac{1}{2}(1+\Gamma^{(10)}_{11})_a{}^b \end{pmatrix};$$

$$t_d = \frac{1}{\sqrt{96}} \begin{pmatrix} 0 & 0 & \sqrt{2}(1+\Gamma^{(10)}_{11})_d{}^b \\ 0 & 0 & \Big[\Gamma^{(10)}_k(1+\Gamma^{(10)}_{11})\Big]_d{}^b \\ \sqrt{2}\Big[(1+\Gamma^{(10)}_{11})C^{(10)}\Big]_{ad} & \Big[(1+\Gamma^{(10)}_{11})\Gamma^{(10)}_l C^{(10)}\Big]_{ad} & 0 \end{pmatrix}$$

As a check, it is easy to verify that

$$C(27) = g^{\boldsymbol{AB}} t_{\boldsymbol{A}} t_{\boldsymbol{B}} = \frac{1}{2} t_{ij} t_{ij} + (C^{(10)})^{ab} t_a t_b + t^2 = \frac{13}{9} = \frac{1}{2}\cdot\frac{78}{27}.$$

The group E_7 contains the maximal subgroup $E_6\times U_1$, with respect to which

$$56\Big|_{E_7} = 27(1) + \overline{27}(-1) + 1(3) + 1(-3)\Big|_{E_6\times U_1};$$
$$133\Big|_{E_7} = 1(0) + 27(-2) + \overline{27}(2) + 78(0)\Big|_{E_6\times U_1}.$$

We see that the representation 56 contains two E_6 singlets with nontrivial U_1 charges. If one of them acquires a vacuum expectation value, then the little group will contain the factor E_6. Let the vacuum expectation value v_7 is acquired by the representation $56(1)$ of the group $E_7\times U_1$, and the corresponding scalar field lies in the representation $1(3)$ of the group $E_6\times U_1 \subset E_7$. Under the $U_1\times U_1$ transformations in

$$E_7\times \underbrace{U_1}_{\beta_1^{(7)}} \supset (E_6\times \underbrace{U_1}_{\beta_2^{(7)}})\times \underbrace{U_1}_{\beta_1^{(7)}}.$$

the vacuum expectation value changes as $v_7 \to \exp\Big(i\beta_1^{(7)} + 3i\beta_2^{(7)}\Big)v_7$. Therefore, it is invariant under the transformations with $\beta_1^{(7)} + 3\beta_2^{(7)} = 0$. Evidently, they constitute the group $U_1\subset U_1\times U_1$.

Let us compare the coupling constants in the original $E_7 \times U_1$ theory and its $E_6 \times U_1$ remnant. The relation between the couplings for the groups E_7 and E_6 is obtained from the comparison of the commutation relations for the generators t_{ij}:

$$t_{ij}\Big|_{E_7} = \frac{1}{\sqrt{2}} t_{ij}\Big|_{E_6} \quad \to \quad e_6 = \frac{e_7}{\sqrt{2}},$$

because

$$A_\mu\Big|_{E_7} = ie_7 A_\mu^{\boldsymbol{A}} t_{\boldsymbol{A}}\Big|_{E_7} \quad \to \quad A_\mu\Big|_{E_6} = ie_6 A_\mu^{\boldsymbol{A}} t_{\boldsymbol{A}}\Big|_{E_6}.$$

To find the coupling constant $e_1^{(6)}$, we write the branching rule for the representation $56(1)$ with respect to the subgroup $E_6 \times U_1 \times U_1$,

$$56(1)\Big|_{E_7 \times U_1} = 27(1,1) + \overline{27}(1,-1) + 1(1,3) + 1(1,-3)\Big|_{E_6 \times U_1 \times U_1}.$$

Therefore, the charge with respect to the little group can be chosen in the form

$$Q_1^{(6)} = \frac{1}{2}\Big(-3Q_1^{(7)} + Q_2^{(7)}\Big),$$

where the normalization is chosen in such a way that it is an integer and the least possible integer.

Earlier we saw that the coupling constant for the U_1 component of $E_7 \times U_1$ is

$$e_1^{(7)} = \frac{e_7}{4\sqrt{3}}.$$

Therefore, the charge $Q_1^{(7)}$ is an eigenvalue of the operator $4\sqrt{3}\, t_1^{(7)}$, where $t_1^{(7)}$ is the generator of the U_1 factor in $E_7 \times U_1$ which is normalized in the same way as the generators of the group E_7.

Let $t\Big|_{U_1\subset E_7}$ be the generator of the U_1 factor in the subgroup $E_6 \times U_1 \subset E_7$ normalized in the same way as all E_7 generators. Then

$$t\Big|_{U_1\subset E_7} = \frac{1}{12}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \mbox{acting on} \quad \begin{pmatrix} 1 \\ 1 \\ 27 \\ \overline{27} \end{pmatrix}.$$

This generator is correctly normalized because

$$\mbox{tr}\left(\left(t\Big|_{U_1\subset E_7}\right)^2\right) = \frac{1}{144}\Big(1\cdot 3^2 + 1\cdot(-3)^2 + 27\cdot 1^2 + 27\cdot 1^2\Big) = \frac{72}{144} = \frac{1}{2}.$$

Due to the branching rule

$$56\Big|_{E_7} = 27(1) + \overline{27}(-1) + 1(3) + 1(-3)\Big|_{E_6 \times U_1},$$

the charge $Q_2^{(7)}$ is an eigenvalue of the operator $12\,t\Big|_{U_1 \subset E_7}$. Therefore, the little group charge is an eigenvalue of the operator

$$\frac{1}{2}\Big(-3\cdot 4\sqrt{3}\,t_1^{(7)} + 12\,t\Big|_{U_1\subset E_7}\Big) = 12\Big(-\frac{\sqrt{3}}{2}t_1^{(7)} + \frac{1}{2}t\Big|_{U_1\subset E_7}\Big).$$

Note that in the right hand side the operator in the brackets is normalized in the same way as the generators of the group E_7. therefore, the coefficient 12 is equal to the ratio of the couplings e_7 and $e_1^{(6)}$,

$$e_1^{(6)} = \frac{e_7}{12}.$$

Thus,

$$e_6 = \frac{e_7}{\sqrt{2}}; \qquad e_1^{(6)} = \frac{e_7}{12} = \frac{e_6}{6\sqrt{2}}$$

or, equivalently,

$$\alpha_6 = \frac{\alpha_7}{2}; \qquad \alpha_1^{(6)} = \frac{\alpha_6}{72}.$$

Next, we construct the branching rule of 248 with respect to the subgroup $\underbrace{E_6 \times U_1}_{\beta_1^{(7)}} \times \underbrace{U_1}_{\beta_2^{(7)}} \subset E_7 \times \underbrace{U_1}_{\beta_1^{(7)}}$,

$$248\Big|_{E_8} = \Big[1(0,0) + 1(2,0) + 1(-2,0)\Big] + \Big[1(0,0) + 27(0,-2) + \overline{27}(0,2)$$
$$+78(0,0)\Big] + \Big[27(1,1) + \overline{27}(1,-1) + 1(1,3) + 1(1,-3)\Big]$$
$$+\Big[27(-1,1) + \overline{27}(-1,-1) + 1(-1,3) + 1(-1,-3)\Big]\Big|_{E_6\times U_1\times U_1}.$$

Calculating the charge with respect to the little group for each term we obtain the decomposition

$$248\Big|_{E_8} = 4\times 1(0) + 2\times 1(3) + 2\times 1(-3) + 2\times 27(-1) + 2\times \overline{27}(1) + 27(2)$$
$$+\overline{27}(-2) + 78(0)\Big|_{E_6\times U_1}.$$

The further symmetry breaking $E_6 \times U_1 \to SO_{10} \times U_1$ can be produced by two different ways. Namely, vacuum expectation values can be acquired by $27(-1)$ or $27(2)$. For the next symmetry breaking steps the number of options becomes larger.

The representations for the further symmetry breaking

The further investigation of the symmetry breaking is made similarly. Vacuum expectation values are acquired by the representations which are present in the branching rules of 248 and contain singlets with respect to the non-Abelian components of the little group with nontrivial U_1 charges:

For the symmetry breaking $E_6 \times U_1 \to SO_{10} \times U_1$

$$27\Big|_{E_6} = 1(4) + 10(-2) + 16(1)\Big|_{SO_{10}\times U_1}.$$

For the symmetry breaking $SO_{10} \times U_1 \to SU_5 \times U_1$

$$16\Big|_{SO_{10}} = 1(-5) + \bar{5}(3) + 10(-1)\Big|_{SU_5\times U_1}.$$

For the symmetry breaking $SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1$

$$10\Big|_{SU_5} = [1,1](6) + [\bar{3},1](-4) + [3,2](1)\Big|_{SU_3\times SU_2\times U_1}.$$

Note that we avoid involving higher representations of various groups present in the symmetry breaking pattern.

The relations between the coupling constants for the non-Abelian groups are obtained by comparing the commutation relations for the corresponding generators using the explicit form of the embeddings.

For instance, the SO_{10} generators $(t_{ij})_{kl} = \frac{i}{2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$ normalized with the metric

$$g_{\boldsymbol{AB}} \to \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}; \qquad g^{\boldsymbol{AB}} \to \frac{1}{4}\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big)$$

satisfy the commutation relations

$$[t_{ij}, t_{kl}] = \frac{i}{2}\Big(\delta_{il}t_{jk} - \delta_{jl}t_{ik} - \delta_{ik}t_{jl} + \delta_{jk}t_{il}\Big).$$

Comparing this equation with the corresponding relation for E_6 we conclude that

$$t_{ij}\Big|_{E_6} = \frac{1}{\sqrt{3}}t_{ij}\Big|_{SO_{10}} \quad \to \quad e_{10} = \frac{e_6}{\sqrt{3}},$$

because in this case

$$A_\mu\Big|_{E_6} = ie_6 A_\mu^{\boldsymbol{A}} t_{\boldsymbol{A}}\Big|_{E_6} \quad \to \quad A_\mu\Big|_{SO_{10}} = ie_{10} A_\mu^{\boldsymbol{A}} t_{\boldsymbol{A}}\Big|_{SO_{10}} = \frac{i}{2}e_{10}\big(A_\mu\big)_{ij} t_{ij}\Big|_{SO_{10}}.$$

Let us construct the embedding $U_5 \subset SO_{10}$. For this purpose we consider a complex 5-component column $z = x + iy$ in the fundamental representation of the group U_5,

$$z \equiv x + iy \to \Omega_5 z = (B + iC)(x + iy) = (Bx - Cy) + i(By + Cx),$$

where the 5×5 matrix $\Omega_5 \in U_5$ was written as the sum of the real part B and the purely imaginary part iC. Note that from the condition $\Omega_5^+\Omega_5 = 1$ we obtain that the real matrices B and C satisfy the constraints

$$B^T B + C^T C = 1; \qquad B^T C = C^T B.$$

The above transformation of z can equivalently be presented as the transformation of a real 10-component column

$$\left(\begin{array}{c} x\\ y \end{array}\right) \to \left(\begin{array}{cc} B & -C\\ C & B \end{array}\right)\left(\begin{array}{c} x\\ y \end{array}\right).$$

From the above constraints on the matrices B and C it is easy to see that the matrix in this equation is orthogonal. Moreover, its determinant is equal to 1 because the U_5 group manifold is connected. Therefore, this matrix belongs to the group SO_{10}.

The properly normalized generators of SO_{10} corresponding to the subgroup SU_5 can be written in the form

$$t_A\Big|_{SU_5\subset SO_{10}} = \begin{cases} \dfrac{1}{\sqrt{2}}\begin{pmatrix} t_{A,5} & 0\\ 0 & t_{A,5}\end{pmatrix} = \dfrac{1}{\sqrt{2}}T(t_{A,5}), & \\ \qquad\qquad\qquad\qquad \text{if } t_{A,5} \text{ is purely imaginary;} & \\ \dfrac{i}{\sqrt{2}}\begin{pmatrix} 0 & t_{A,5}\\ -t_{A,5} & 0\end{pmatrix} = \dfrac{1}{\sqrt{2}}T(t_{A,5}), & \\ \qquad\qquad\qquad\qquad \text{if } t_{A,5} \text{ is real,} & \end{cases}$$

where $t_{A,5}$ (with $A=1,\ldots,24$) are the generators of the SU_5 fundamental representation normalized by the condition

$$\mathrm{tr}\big(t_{A,5}t_{B,5}\big) = \frac{1}{2}\delta_{AB}.$$

(Certainly, it is convenient to choose them in such a way that they are either real or purely imaginary). Due to the factor $1/\sqrt{2}$ coming from the normalization condition, the coupling constants for the group SO_{10} and its SU_5 subgroup are related by the equation

$$e_5 = \frac{e_{10}}{\sqrt{2}}.$$

We also see that the properly normalized generator of the U_1 subgroup is

$$t\Big|_{U_1\subset SO_{10}} = -\frac{i}{\sqrt{20}}\left(\begin{array}{cc} 0 & -1_5 \\ 1_5 & 0 \end{array}\right).$$

Similarly, the last embedding $SU_3\times SU_2\times U_1\subset SU_5$

$$\omega_5 = \left(\begin{array}{cc} e^{-2i\beta_2^{(5)}}\omega_3 & 0 \\ 0 & e^{3i\beta_2^{(5)}}\omega_2 \end{array}\right)$$

gives

$$e_2 = e_3 = e_5.$$

Thus, for the coupling constants corresponding to the non-Abelian groups we obtain the relations

$$e_2 = e_3 = e_5 = \frac{e_{10}}{\sqrt{2}} = \frac{e_6}{\sqrt{6}} = \frac{e_7}{\sqrt{12}} = \frac{e_8}{\sqrt{60}},$$

which can equivalently be rewritten in the form

$$\alpha_2 = \alpha_3 = \alpha_5 = \frac{\alpha_{10}}{2} = \frac{\alpha_6}{6} = \frac{\alpha_7}{12} = \frac{\alpha_8}{60}.$$

For the symmetry breaking $G\times U_1 \to H\times U_1$ the coupling constants for the U_1 groups are calculated according to the following algorithm:

1. It is necessary to construct decomposition of the representation which acquires the vacuum expectation value with respect to the subgroup $H\times U_1\times U_1\subset G\times U_1$.

2. Next, one should find the expression for the little group charge. At all steps except for the last one it is chosen in such a way that this charge takes minimal possible integer values. At the last step the charge normalization is chosen so that the number of MSSM representations in the decomposition of 248 with the U_1 charges coinciding with the hypercharge will be as much as possible.

3. After this, we construct the generators of the group $U_1\times U_1$ normalized in the same way as the generators of the group G.

4. Then we construct the generator of the little group and extract from it the operator normalized in the same way as the generators of the group G. The coefficient before it gives the ratio $e_G/e_1^{(H)}$.

With the help of this algorithm for each option of the symmetry breaking we obtain a sequence of the U_1 charges. For each of them finally we calculate $\mathrm{tg}\,\theta_W = e_1^{(Y)}/e_2$.

Options for the further symmetry breaking

Option	$E_6 \times U_1$	$SO_{10} \times U_1$	$SU_5 \times U_1$	$\sin^2\theta_W$
B-1-1-1	$27(-1)\Big\vert_{E_6\times U_1}$	$16(-1)\Big\vert_{SO_{10}\times U_1}$	$10(-1)\Big\vert_{SU_5\times U_1}$	3/8
B-1-1-2	$27(-1)\Big\vert_{E_6\times U_1}$	$16(-1)\Big\vert_{SO_{10}\times U_1}$	$10(4)\Big\vert_{SU_5\times U_1}$	3/5
B-1-2-1	$27(-1)\Big\vert_{E_6\times U_1}$	$16(3)\Big\vert_{SO_{10}\times U_1}$	$10(-2)\Big\vert_{SU_5\times U_1}$	3/5
B-1-2-2	$27(-1)\Big\vert_{E_6\times U_1}$	$16(3)\Big\vert_{SO_{10}\times U_1}$	$10(3)\Big\vert_{SU_5\times U_1}$	3/4
B-2-1-1	$27(2)\Big\vert_{E_6\times U_1}$	$16(1)\Big\vert_{SO_{10}\times U_1}$	$10(-2)\Big\vert_{SU_5\times U_1}$	3/5
B-2-1-2	$27(2)\Big\vert_{E_6\times U_1}$	$16(1)\Big\vert_{SO_{10}\times U_1}$	$10(3)\Big\vert_{SU_5\times U_1}$	3/4

We see that the option B-1-1-1 is the only one which gives the correct value of the Weinberg angle. It is interesting that this option corresponds to the minimal possible absolute values of the U_1 charges.

Group	Quantum numbers						Couplings
E_8	248						e_8
$E_7 \times U_1$	$1(0)$	$1(2)$	$1(-2)$	$133(0)$	$56(1)$	$56(-1)$	$e_7 = e_8/\sqrt{5}$ $(e_1^{(7)} = e_7/4\sqrt{3})$
$E_6 \times U_1$	$1(0)$	$1(-3)$	$1(3)$	$78(0)$ $1(0)$ $27(-1)$ $\overline{27}(1)$	$27(-1)$ $\overline{27}(-2)$ $1(0)$ $1(-3)$	$\overline{27}(1)$ $27(2)$ $1(0)$ $1(3)$	$e_6 = e_7/\sqrt{2}$ $(e_1^{(6)} = e_6/6\sqrt{2})$
$SO_{10} \times U_1$	$9 \times 1(0) + 3 \times 1(4) + 3 \times 1(-4) + 45(0) + 3 \times 10(-2)$ $+3 \times 10(2) + 3 \times 16(-1) + 3 \times \overline{16}(1) + 16(3) + \overline{16}(-3)$						$e_{10} = e_6/\sqrt{3}$ $(e_1^{(10)} = e_{10}/4\sqrt{3})$
$SU_5 \times U_1$	$16 \times 1(0) + 4 \times 1(5) + 4 \times 1(-5) + 24(0) + 6 \times 5(2)$ $+6 \times \bar{5}(-2) + 4 \times 5(-3) + 4 \times \bar{5}(3) + 10(4) + \overline{10}(-4)$ $+4 \times 10(-1) + 4 \times \overline{10}(1)$						$e_5 = e_{10}/\sqrt{2}$ $(e_1^{(5)} = e_5/2\sqrt{10})$
$SU_3 \times SU_2 \times U_1$	$25 \times [1,1](0) + 5 \times [1,1](1) + 5 \times [1,1](-1)$ $+[1,3](0) + 10 \times [1,2](1/2) + 10 \times [1,2](-1/2)$ $+10 \times [3,1](-1/3) + 10 \times [\bar{3},1](1/3) + 5 \times [3,1](2/3)$ $+5 \times [\bar{3},1](-2/3) + 5 \times [3,2](1/6) + 5 \times [\bar{3},2](-1/6)$ $+[3,2](-5/6) + [\bar{3},2](5/6) + [8,1](0)$						$e_3 = e_2 = e_5$ $(e_1^{(Y)} = e_5\sqrt{3/5})$ $\sin^2\theta_W = 3/8$

Wonderfully, this option is the only one which contains all representations needed for the accommodation of all chiral MSSM superfields.

Details of the symmetry breaking for the options B-1-1-2, B-1-2-1, and B-2-1-1

The options B-1-1-2, B-1-2-1, and B-2-1-1 lead to the same value of the Weinberg angle $\sin^2\theta_W = 3/5$ and to the same branching rule of the representation 248 with respect to $SU_3\times SU_2\times U_1$,

$$\begin{aligned}
248\Big|_{E_8} &= 19\times[1,1](0) + 8\times[1,1](1/2) + 8\times[1,1](-1/2) + 12\times[1,2](0)\\
&+4\times[1,2](1/2) + 4\times[1,2](-1/2) + [1,3](0) + 8\times[3,1](1/6)\\
&+8\times[\bar 3,1](-1/6) + 6\times[3,1](-1/3) + 6\times[\bar 3,1](1/3)\\
&+[3,1](2/3) + [\bar 3,1](-2/3) + 2\times[3,2](-1/3) + 2\times[\bar 3,2](1/3)\\
&+4\times[3,2](1/6) + 4\times[\bar 3,2](-1/6) + [8,1](0)\Big|_{SU_3\times SU_2\times U_1}.
\end{aligned}$$

In this case the representation $[1,1](-1)$ needed for the superfields corresponding to the right charged leptons is absent. Therefore, these options are not acceptable for phenomenology.

The options B-1-2-2 and B-2-1-2 lead to the same value of the Weinberg angle $\sin^2\theta_W = 3/4$ and to the same branching rule for the representation 248 with respect to $SU_3\times SU_2\times U_1$,

$$\begin{aligned}
248\Big|_{E_8} &= 17\times[1,1](0) + 9\times[1,1](1/3) + 9\times[1,1](-1/3) + [1,2](1/2)\\
&+[1,2](-1/2) + 9\times[1,2](1/6) + 9\times[1,2](-1/6) + [1,3](0)\\
&+9\times[3,1](0) + 9\times[\bar 3,1](0) + 3\times[3,1](1/3) + 3\times[\bar 3,1](-1/3)\\
&+3\times[3,1](-1/3) + 3\times[\bar 3,1](1/3) + 3\times[3,2](-1/6)\\
&+3\times[\bar 3,2](1/6) + 3\times[3,2](1/6) + 3\times[\bar 3,2](-1/6)\\
&+[8,1](0)\Big|_{SU_3\times SU_2\times U_1}.
\end{aligned}$$

In this case there are no representations $[1,1](-1)$ needed for the superfields corresponding to the right charged leptons and no representations $[3,1](2/3)$ corresponding to the right upper quarks. Therefore, all these options are not acceptable for phenomenology.

Conclusion

- Using the group theory we analyzed a possibility of the symmetry breaking pattern

  $$E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \times U_1 \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1$$

  provided that only parts of the representation 248 can acquire vacuum expectation values. Also we assume that all U_1 groups in the considered chain are different.

- Among 6 different options for the symmetry breaking there is the only one which leads to the correct value of the Weinberg angle and produces all representations needed for the accommodation of all MSSM chiral superfields. This option corresponds to the minimal absolute values of all U_1 charges of the fields responsible for the symmetry breaking.

- Presumably, the considered symmetry breaking pattern could allow to understand how a low-energy chiral theory appears.

Conclusion

- The representation 248 of the group E_8 is fundamental and adjoint simultaneously. Moreover, more than one 248 representations are needed, so that ther is an interesting possibility to use the finite $\mathcal{N}=4$ SYM with the group E_8 for the Grand Unification.
- We did not study dynamics of the considered symmetry breaking pattern and made the investigation only using the group theory methods.
- Some other similar symmetry breaking patterns like

$$E_8 \to E_7 \times U_1 \to E_6 \times U_1 \to SO_{10} \to SU_5 \times U_1 \to SU_3 \times SU_2 \times U_1$$

can also be considered.

Thank you for the attention!

P.S. All group theory relations were taken from

$\boxed{\text{R. Slansky, Phys. Rept. } \mathbf{79} \text{ (1981), 1.}}$