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REPORT

Primordial black holes abundance from the perturbations in the inflaton model

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1 Introduction

Primordial black holes have been receiving increasing interest for research in recent years, especially after the discovery of the long-sought gravitational waves from the merger of binary black holes.[1]. Their influence on the cosmology is really big. Some researchers suggest them as possible sources of gravitational waves. Attempts to detect the predicted Hawking radiation are also a point of interest. In addition, primordial black holes are considered in a number of ways as objects that can explain the inflationary model of the development of the Universe.

There are various models explaining how exactly PBHs could have formed in the early epoch of the Universe [2, 3, 4]. It follows from these models that, unlike "ordinary"black holes, PBHs can potentially have a very wide spread of masses, literally from the mass of the order of an elementary particle to the order of a hundred solar masses.

However, despite this situation nowadays, PBHs should have been more massive in the past. Thus, according to calculations, PBHs with mass $M < 10^{15}$ g by now should have already evaporated due to Hawking radiation.

To estimate the number of PBHs, a numerical parameter f was introduced, which is defined as their fraction of the total mass of dark matter at the present time. If $f < 10^{-3}$, we can confidently assert that PBHs should not be considered as candidates for the role of dark matter [4].

In this report, there will be some calculations presented that make it possible to estimate the emission spectrum of PBHs, as well as the f parameter, based on single-field inflation models, with one or two perturbations in the inflaton potential. It should be understood that the f parameter, generally speaking, depends on many factors, such as the radiation range under consideration, the threshold of density contrast, as well as the method used to calculate the PBH mass fraction. In this essay, the main attention will be paid to the last factor - the calculation of the PSH mass fraction in peak theory [5, 6, 7, 8, 9].

The peak theory of Gaussian random fields was initiated in [6]. Following this, a group of researchers calculated the parameter f using a simplified model [7]; their result is in good agreement with the Press — Schechter theory [9], at least for the emission spectrum.

The main idea proposed by peak theory is to try to calculate the density of peaks in regions where compact objects are expected to form, such as galaxies or PBHs in the form of their active nuclei. Then compare the result with the known radiation background. These local "densities" are local maxima of the radiation field, so that an analytical joint probability distribution function (PDF) can be constructed, and then its first and second derivatives can be calculated, from which it will be possible to determine the desired peaks.

2 Preliminary calculations

2.1 Basic equations

Consideration should start with the single-field inflation model, where inflation is explained by a scalar inflaton field not related to gravity, its effect is

$$S = \int \sqrt{-g}(\frac{m_P^2}{2}R - 0.5\partial_\mu\phi\partial^\mu\phi - V(\phi)), \tag{1}$$

Where ϕ is the inflation field, $V(\phi)$ is its potential, R is the Ricci scalar, and m_P is the Planck mass.

During the inflationary era, it is more convenient to measure cosmological expansion in terms of the number of e-folds N instead of time. It can be defined as the time integral of the Hubble constant.

We will also introduce two parameters that will further show the state of the inflaton field:

$$\epsilon = -\frac{\dot{H}}{H^2}, \qquad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{2}$$

In the slow-roll inflation model, these parameters are always much less than 1. However, if we consider the ultra slow-roll inflation model, the values of ϵ and η can vary over a very wide range in different situations. Let us write the Klein-Gordon equation for a homogeneous and isotropic Universe in terms of ϵ, η, and N

$$\phi_{,NN} + (3-\epsilon)\phi_{,N} + \frac{1}{H^2}V_{,\phi} = 0, \tag{3}$$

and also the Friedmann equation

$$H^2 = \frac{V}{3-\epsilon}m_P^2. \tag{4}$$

In the model under consideration, we will talk about space-time, so it also makes sense to write the equation for the interval $ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j$. Also, to calculate the value of the f parameter, the R parameter is useful — primordial curvature perturbation,

$$R = \Phi + \frac{H}{\dot{\phi}}\delta\phi. \tag{5}$$

The equation of motion for R can be written using the Mukhanov – Sasaki equation [10]:

$$R_{k,NN} + (3+\epsilon-2\eta)R_{k,N} + \frac{k^2}{H^2 e^2 N}R_k = 0. \tag{6}$$

Finally, the parameters ϵ and η can be expressed in terms of N:

$$\epsilon = \frac{\phi_{,N}^2}{2m_P^2}, \qquad \eta == \frac{\phi_{,N}^2}{2m_P^2} - \frac{\phi_{,NN}}{\phi_{,N}}. \tag{7}$$

After solving the equations presented here, it will be possible to obtain an expression for the spectrum of the radiation intensity and accordingly the mass fraction of PBHs.

2.2 Calculation of the emission spectrum

As mentioned in the introduction, the radiation spectrum can be obtained using the radiation density distribution function, we denote it as $P_\delta(k)$. In the radiation-dominated era, $P_\delta(k)$ can be further related to the dimensionless power spectrum of primordial curvature perturbation $P_R(k)$ [7]:

$$P_\delta(k) = \frac{16}{81}(\frac{k}{aH})^4 P_R(k), \tag{8}$$

$$P_R(k) = \frac{k^3}{2\pi^2}|R_k|^2, \quad k << aH. \tag{9}$$

Within the slow-roll inflation model we can also write $P_R(k)$ in a different form: $P_R(k) = A_s(k/k_*)^{n_s-1}$, where n_s is the scalar spectral index, A_s is the amplitude, with the central values being $n_s = 0.965$, and $A_s = 2.10 \cdot 10^{-9}$ on the CMB pivot scale, $k_* = 0.05\text{Mpc}^{-1}$ [11]. Limitations on $P_R(k)$ in the range where signals from the PBHs are expected are given in [2, 3, 4].

To find the local maxima of the function it is necessary to ensure its differentiability over the entire space, as well as to avoid divergence at large distances. This can be achieved by using convolution

$$\delta(\mathbf{x}, R) = \int d^3x' W(\mathbf{x} - \mathbf{x}', R)\delta(\mathbf{x}'). \tag{10}$$

Here $W(\mathbf{x}, R)$ is a window function. We will use the normal distribution as the window function, after the Fourier transform it will look like $W_{Fourier}(k, R) = e^{-k^2R^2/2}$, so

$$W(\mathbf{x}, R) = \frac{e^{-x^2/2R^2}}{V(R)}, \tag{11}$$

$V(R) = (\sqrt{2\pi}R)^3$ is a normalization factor[12].

After the smoothing procedure, you can write an expression for the density contrast variance in the R range:

$$\sigma_\delta^2(R) = \int\limits_0^\infty k^{-1} W^2_{Fourier}(k, R) P_\delta(k) dk. \tag{12}$$

From the homogeneity and isotropy of space (and, therefore, background radiation), it unambiguously follows that $\sigma_\delta^2(R)$ does not depend on the radius vector. Similarly, the i-th spectral moment of the smoothed density contrast is defined as

$$\sigma_i^2(R) = \int\limits_0^\infty k^{2i-1} W^2_{Fourier}(k, R) P_\delta(k) dk = 16/81 \int\limits_0^\infty k^{2i-1} W_{Fourier}(k, R)(kR)^4 P_R(k), \tag{13}$$

Where i is natural number and $\sigma_0 = \sigma_\delta$.

2.3 PBH mass and abundance

There is a Carr-Hawking model of collapse [13], in which the relation between the PBH mass and the horizon mass at the moment of its formation is derived

$$M = \kappa M_H = \frac{\kappa}{2GH}. \tag{14}$$

M_H is the horizon mass, κ is the efficiency of collapse. Considering that we are talking about a radiation-dominated era, the Hubble constant is equal $H = 1/(2t)$, which means that the PBH mass is $M = \kappa t/G$.

Cosmological expansion is an adiabatic process, so in accordance with [2] we get the estimate M

$$\frac{M}{M_{SUN}} = 1.13 \cdot 10^{15} \frac{\kappa}{0.2} (\frac{g_*}{106.75})^{-1/6} (\frac{k_*}{k_{PBH}})^2. \tag{15}$$

where g_* is the effective number of relativistic degrees of freedom of energy density, and $k_{PBH} = 1/R$ is the wave number of the PBH that exits the horizon. The values for κ and g_* are taken from [14]. Having equation (15), we can rewrite expression (13) by expressing the variance in terms of the PBH mass.

Let us denote by $\beta(M)$ the mass fraction of PBHs at the moment of their appearance:

$$\beta(M) = \frac{\rho_{PBH}(M)}{\rho_R}, \quad t = t_{PBH formation}. \tag{16}$$

In this case, the f parameter is currently equal to

$$f(M) = \frac{\rho_{PBH}(M)}{\rho_{DM}}, \quad t = today. \tag{17}$$

Here we do not take into account such factors as radiation, accretion, and merging of PBHs.

Finally, in accordance with [2] $f(M)$ and $\beta(M)$ are proportional, so [2]

$$f(M) = 1.68 \cdot 10^8 (M/M_{SUN})^{-0.5} \beta(M). \tag{18}$$

3 Final model

The sequence of actions for calculating quantities that are interesting to us can be represented as follows:

$$\delta V(\phi) \to P_R(k) \to \sigma_i^2(M) \to \beta(M) \to f(M). \tag{19}$$

In the previous chapter, the expressions for each of the indicated transitions were deduced. It remains to express the final parameter in terms of the initial numbers so that we can compose the final model.

We will express the magnitude of the probable disturbance as follows:

$$\delta V(\phi) = -A(\phi - \phi_0)F(\frac{\phi - \phi_0}{\sqrt{2}\sigma}), \tag{20}$$

where F is an even function satisfying the relation $\lim_{x\to\infty} xF(x) = 0$. Thus, the final model will have three variable parameters characterizing the shape and location of the disturbance: A, ϕ_0 and σ. It should be taken into account that if A is small enough, there will be a flat surface near ϕ_0 - an extremely small perturbation indistinguishable from the background. This should be taken into account in the model. Also, for the sake of simplicity and universality, $F(x)$ will be taken as the Lorentz function.

Let's deal with the background level. It can be taken as Kachru – Kallosh – Linde – Trivedi potential [15],

$$V_b(\phi) = V_0\frac{\phi^2}{\phi^2 + (m_P/2)^2}. \tag{21}$$

Again according to [15] $V_0/m_P^4 = 10^{-10}$, so for large k the form of the radiation spectrum $P_R(k)$ can retain scale invariance and a relatively small tensor-to-scalar ratio is favored by the CMB observations [11]. Since, as mentioned above, $F(x)$ is taken as the Lorentz function,

$$\delta V(\phi) = -A\frac{\phi - \phi_0}{1 + (\phi - \phi_0)^2/(2\sigma^2)}. \tag{22}$$

The graph of the function $V(\phi) = V_b(\phi) + \delta V(\phi)$ is presented on the Fig. 1.[16]

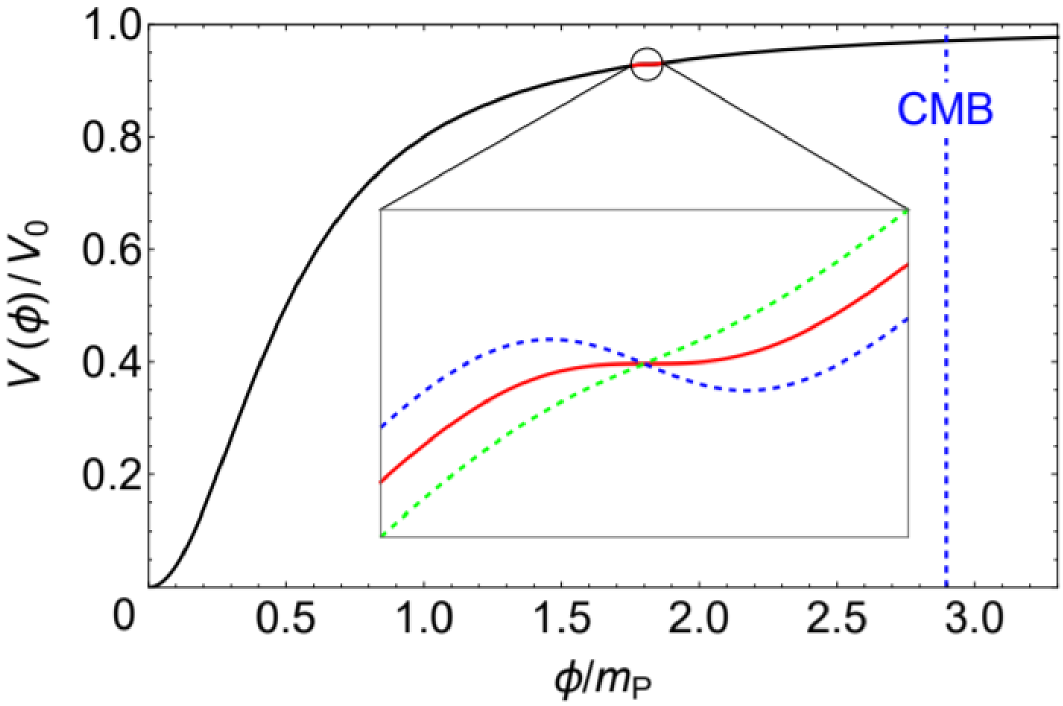

Fig 1 – The background inflaton potential $V_b(\phi)$ and the plateau (i.e., the USR region) caused by the perturbation $\delta V(\phi)$, with $\phi_0/m_P = 1.81$ and $\sigma/m_P = 0.057$. For comparison, A is chosen to be $0.5V_{b,\phi}(\phi_0)$ (green dashed line), $V_{b,\phi}(\phi_0)$ (red line), and $1.5V_{b,\phi}(\phi_0)$ (blue dashed line), respectively. The CMB pivot scale k_* corresponds to $\phi_0/m_P = 2.90$.

Fig. 2 shows the dependence of the $\delta V(\phi)$ on the $P_R(k)$. Shown are two graphs representing the numerical solutions of equations (6) and (8) - (9), respectively. The red line shows the scale invariant plot, just for clarity and comparison.[16]

Now, having the dependence $P_R(k)$, we set the requirement for the number of PBHs $f(M) \sim 0.1$ in the mass ranges of interest to us: $10^{-17}M_{SUN}$, $10^{-13}M_{SUN}$,

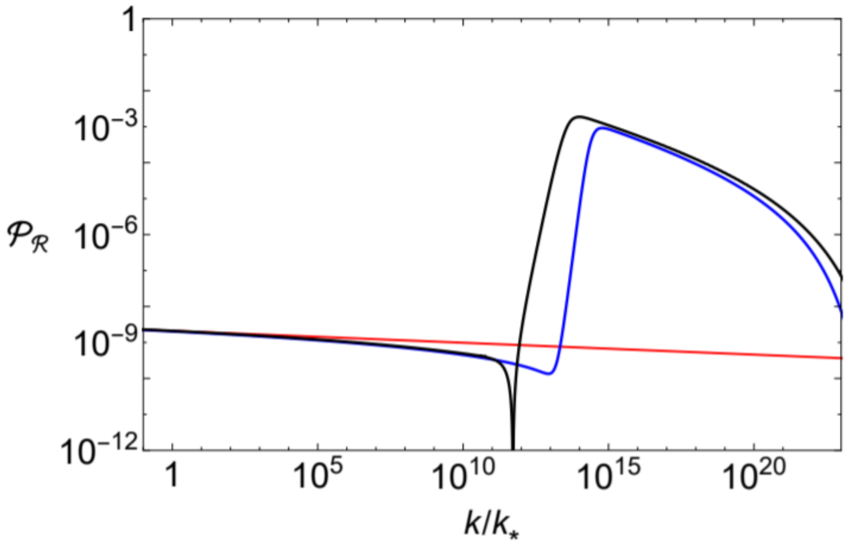

Fig 2 – The power spectra $P_R(k)$ influenced by the perturbation $\delta V(\phi)$: the power spectrum by numerically solving Eq. (6) (black line), the power spectrum from the SR approximations in Eq. (8-9) (blue line), and the nearly scale-invariant power spectrum for comparison (red line).

$30M_{SUN}$. We have three variable parameters in the model — A, ϕ_0 and σ, but there are only two boundary conditions — for M and for f. To avoid divergence during model operation, we add one more condition for $A = V_{b,\phi}(\phi_0)$. By equating it to the background level, we will ensure that our indignation reaches a plateau around ϕ_0.

Thus, we get rid of one variable. Now the task is simplified and comes down to just determining the values of the two remaining parameters, that are ϕ_0 and σ.

M/M_{SUN}	ϕ_0/m_P	σ/m_P
10^{-17}	1.31	0.0831881
10^{-17}	1.81	0.0405471
30	2.56	0.0159387

Table 1. The parameters ϕ_0 and σ for the PBH abundances $f \sim 0.1$ in the three typical mass windows at $10^{-17}M_{SUN}$, $10^{-13}M_{SUN}$, and $30M_{SUN}$,

respectively. With M increasing, ϕ_0 increases (the USR stage occurs earlier) and σ decreases (the duration of the USR stage shortens). Moreover, f is highly sensitive to σ.

The spectra $P_R(k)$ and corresponding $f(M)$ are shown on Fig. 3. Corresponding values for ϕ_0 and σ can be seen in Tab. 1.

Looking at these picture and table we now can sum up the results. The first conclusion is that the more PBH mass M grows the larger value P_R peak gets. Actually, this satisfies (15) and comes along with the theory, because large M means a small k_{PBH}. Also a smaller k_{PBH} corresponds to an earlier time, so M and ϕ_0 grow simultaneously. Second conclusion is that a larger ϕ_0 would enhance P_R, so if the height of P_R maintains around 10^{-2} , the parameter σ must decrease accordingly. This is because the earlier the USR inflation occurs (with larger ϕ_0), the more slowly the inflaton rolls down.

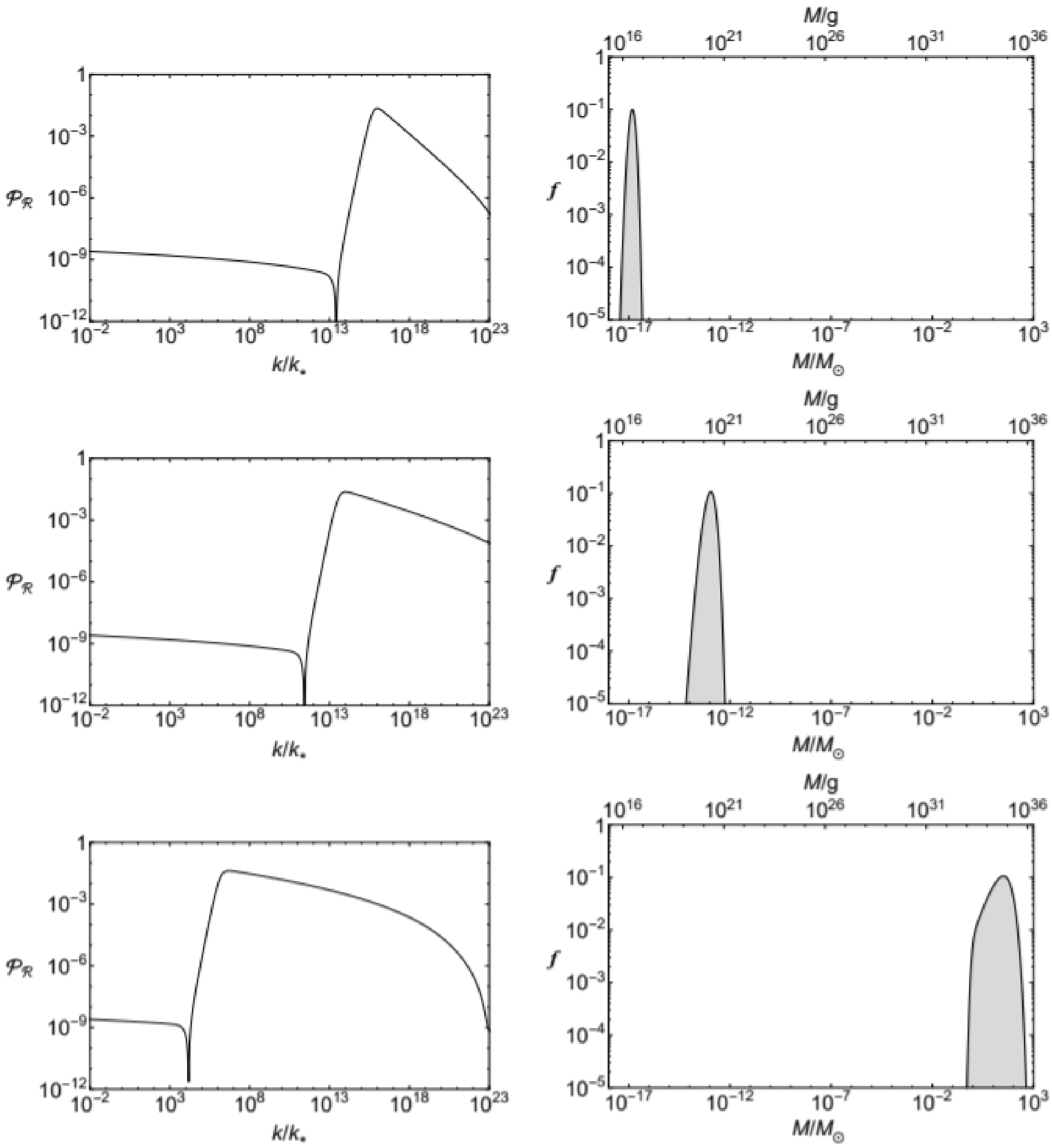

Fig 3 – The power spectra PR(k) and the relevant PBH abundances f(M) with the PBH masses M in the three typical mass windows at $10^{-17}M_{SUN}$, $10^{-13}M_{SUN}$, and $30M_{SUN}$, respectively. If $f \sim 0.1$, P_R needs to be enhanced up to at least 10^{-2} on small scales, seven orders of magnitude higher than its value on large scales. From (15), with M increasing, k_{PBH} decreases, so the peak of $P_R(k)$ moves to larger scales.

4 Conclusion

The study of PBHs is an interesting combination of black hole physics and cosmology. This was largely aided by the discovery of merging gravitational waves from binary black holes in the recent years. One of the main motivations for studying PBH is to find an effective dark matter candidate. Currently, compared to WIMP-like or axion-like DM candidates, experimental limitations on PBHs are still rather weak. Therefore, the purpose of works that were considered here is to study the content of PBHs by considering disturbances in inflaton potential.

Summing up the results of this report, a number of conclusions should be made.

First. The perturbation shape can be described by a fairly simple model of three parameters, which makes it possible to construct a background radiation scheme as a single analytical function.

Second. It is shown that for masses of primordial black holes in 10^{-17}, 10^{-13} and 30 solar masses, a state of dark matter can be reached, at which the mass fraction of primordial black holes is about 10

It should also be remembered that the calculated parameter f depends not only on the values considered. As noted above, the described model is rather primitive and does not take into account a number of factors. Generally speaking, restrictions on the mass fraction of PBHs in dark matter are discussed very widely in the scientific community [17, 18, 19].

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