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DEPARTMENT №40 «PARTICLE PHYSICS»

ABSTRACT ON COSMOPARTICLE PHYSICS

MAGNETIC MONOPOLES IN GRAND UNIFIED THEORIES

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Introduction

«One would be surprised if Nature had made no use of it» — with these words Paul Dirac concluded his pioneering paper on existence of magnetic monopoles back in 1931 [1]. Indeed, Maxwell's equations without a source represent an amazing beauty of nature, because these equations are perfectly symmetrical with respect to the electric and magnetic fields. Opposite, Maxwell's equations with source are not symmetrical anymore — there is no isolated source of magnetic field. It is with this motivation that Paul Dirac was conducted his research on monoples, and came up with surprising result — the existence of a magnetic monopole inevitably leads to the quantization of electric charge!

But with the development of particle physics, the quantization of the charge is no longer surprising — now physicists are certain about the existence of particles with a non-integer charge. Forty years after Dirac, physicists came up to unified theories of strong and electroweak interactions, one of major motivation (beside symmetry of course) for them is non-integer charge, which naturally arise in Georgi-Glashow theory [2]. But beside that, this model contains one more amazing feature — magnetic monopoles. 't Hooft [3] and Polykov [4] shown that theories with unified gauge field inevitably contain magnetic monopole and furthermore, such monopole is not an elementary particle is a topological defect and the quantization condition depend on its topology configuration!

Exactly from the researches of 't Hooft and Polykov the modern theory of magnetic monopoles are started. These magnetic monopoles would be generated after spontaneous symmetry breaking in any Grand Unified Theory (GUT) which contains a compact group.

In the history of theoretical physics, the hypothesis about the possible existence of a magnetic monopole has no analogy. There is no other purely theoretical construction that has managed not only to survive, without any experimental evidence, in the course of more than a century, but has also remained the focus of intensive research by generations of physicists.

1. Dirac Magnetic Monopole

A magnetic monopole is a hypothetical particle with only one isolated magnetic pole. Pole Dirac in 1931 [1] proposed such particle with magnetic charge g (by analogy with electric charge), and developed a theory of quantization of both electric and magnetic charges.

According to Dirac, if the magnetic monopole exist, then it's field strength $\mathbf{H}$ is written as:

$$\mathbf{H} = \frac{g}{4\pi}\frac{\mathbf{r}}{r^3}, \tag{1.1}$$

where r is distance from the source of magnetic field, i.e. monopole and $\mathbf{r}$ — is accordingly distance vector.

Those magnetic field contradicts to classical electrodynamics, because the Maxwell equation now is changed as:

$$\nabla \cdot \mathbf{H} = \frac{g}{4\pi}\left(\frac{\nabla \cdot \mathbf{r}}{r^3} + \mathbf{r} \cdot \nabla \frac{1}{r^3}\right) = 0, \tag{1.2}$$

this is true only in a region $r \neq 0$, but if $r = 0$ one need to consider a flux of magnetic field through a sphere with infinity radii:

$$\int \mathbf{ds} \cdot \mathbf{H} = \int \mathbf{ds} \cdot \frac{g}{4\pi}\frac{\mathbf{r}}{r^3} = \int ds \frac{g}{4\pi r^2} = g, \tag{1.3}$$

and now the magnetic field divergence is written as:

$$\nabla \cdot \mathbf{H} = g\delta^3(\mathbf{r}). \tag{1.4}$$

According to this equation, deviation from electrodynamics is observed only in local point (monopole itself), but in quantum theory it corresponds to large energy, where classic electrodynamics doesn't work (it will be shown further).

Let us now imagine magnetic field 1.4 in the form of vector-potential $\mathbf{A}$:

$$\mathbf{H} = \nabla \times \mathbf{A}. \tag{1.5}$$

To solve this system in terms of $\mathbf{A}$, let us transform those equations in spherical coordinate system:

$$\frac{g}{4\pi r^2}\mathbf{e_r} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{e_r} & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\varphi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}, \tag{1.6}$$

where $\mathbf{e_r}$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ — basis in spherical system. Let us transform it into system of equations:

$$\begin{cases} \dfrac{g}{4\pi r^2} = \dfrac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - \dfrac{\partial}{\partial\varphi}(rA_\theta), \\ 0 = \dfrac{\partial}{\partial r}(r\sin\theta A_\varphi) - \dfrac{\partial A_r}{\partial\varphi}, \\ 0 = \dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{\partial A_r}{\partial\theta}. \end{cases} \tag{1.7}$$

According to equation 1.4, rotor of vector-potential equal to 0 not everywhere — exception is only one point ($r = 0$), i.e. that or solves for $\mathbf{A}$ don't exist at all, or there are infinitely many solutions with correction on gradient of some function. Because of infinitely many solutions, let us consider some extraction conditions $A_r = 0$ and $A_\theta = 0$, after that the third equation from system 1.7 is true automatic. Solving the first equation, we become to:

$$A_\varphi = \frac{g}{4\pi r\sin\theta}(C - \cos\theta), \tag{1.8}$$

as it is clearly seen, $r\sin\theta A_\varphi$ doesn't depend on r, which mean, that second equation is also true automatic. Then vector-potential is written as:

$$\mathbf{A} = \frac{g}{4\pi r\sin\theta}(C - \cos\theta)\mathbf{e}_\varphi. \tag{1.9}$$

From formula 1.11 seen that $\mathbf{A}$ is singular at $\theta = 0$ and $\theta = \pi$ or along Z-axis in rectangular coordinate system. But we can reduce singularity of $\mathbf{A}$ with variation on C.

If $C = 1$:

$$\begin{cases} \mathbf{A}_N = \dfrac{g}{4\pi r \sin\theta}(1 - \cos\theta)\mathbf{e}_\varphi, \\ \mathbf{A}_S \to \infty, \end{cases} \tag{1.10}$$

if C = -1:

$$\begin{cases} \mathbf{A}_S = -\dfrac{g}{4\pi r \sin\theta}(1 + \cos\theta)\mathbf{e}_\varphi, \\ \mathbf{A}_N \to \infty, \end{cases} \tag{1.11}$$

where $\mathbf{A}_S$ and $\mathbf{A}_N$ — vector-potentials of «north» and «sought» directions. Half-axis on which $\mathbf{A} \to \infty$ called Dirac string (see fig. 1.1).

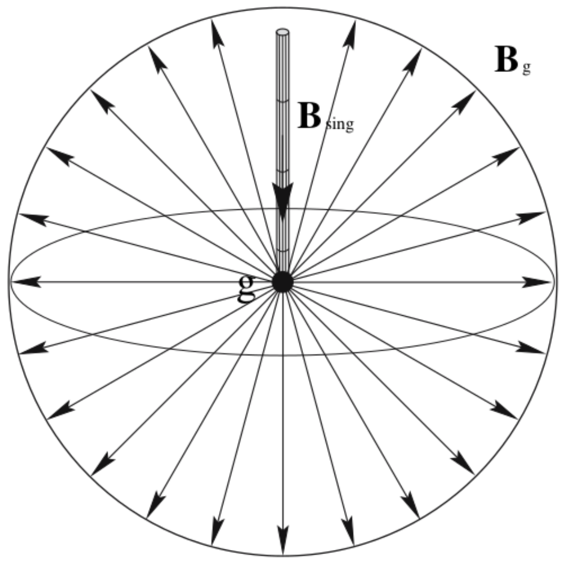

Fig. 1.1 — Dirac String (singularity along Z axis)

Since that $\mathbf{A}_S$ and $\mathbf{A}_N$ describe the same equation 1.5, they have to differ by gradient of some function α:

$$\mathbf{A}_N - \mathbf{A}_S = \nabla\alpha, \tag{1.12}$$

a linear function on φ correctly fit that equation:

$$\alpha = \frac{g}{2\pi}\varphi. \tag{1.13}$$

The 2π in determination correspond to rotation of Dirac string from $\mathbf{A}_N$ to $\mathbf{A}_S$, of course one can determine the α that it would rotate Dirac string at any direction. In addition to the expected Coulomb field we obtain a singular flux of the magnetic field. The very important point is that this extra piece resolves the above-mentioned paradox with the Maxwell equations, because the flux of the string field exactly cancels the contribution from the Coulomb-like part and a total flux of the fields through the closed surface with a monopole inside is 0.

Let's now take a look at theory with scalar field ϕ. A gauge transformation is looked like:

$$\begin{cases} A_\mu \to A_\mu - \partial_\mu \alpha, \\ \phi \to \phi e^{-ie\alpha}, \end{cases} \tag{1.14}$$

let those gauge transformation transform $\mathbf{A}_N$ into $\mathbf{A}_S$, then scalar field should not change:

$$\begin{cases} \phi' = \phi e^{-ie\alpha}, \\ \phi(\varphi = 0) = \phi(\varphi = 2\pi), \\ \phi'(\varphi = 0) = \phi'(\varphi = 2\pi). \end{cases} \tag{1.15}$$

Then:

$$e^{-ie\alpha(\varphi=0)} = e^{-ie\alpha(\varphi=2\pi)}, \tag{1.16}$$

and from there:

$$\alpha(\varphi = 0) - \alpha(\varphi = 2\pi) = \frac{2\pi n}{e}. \tag{1.17}$$

Finally, we get Dirac's quantization:

$$g \cdot e = 2\pi n. \tag{1.18}$$

According to the obtained results, the symmetrization of Maxwell's equation as expected leads to existence of magnetic monopoles. But with presence of magnetic monopole two non-trivial effects arise:

1) if magnetic monopoles exist in the universe, then all electric charge in the universe must be quantized according to Dirac quantization condition 1.18;
2) and such symmetrization leads to singularity in magnetic field of the monopole — Dirac string, but the direction of Dirac string is changing correspond to gauge transformations, this shows that Dirac strings are not gauge invariant, which is consistent with the fact that they are not observable.

2. Non-Abelial Gauge theories: 't Hooft–Polyakov monopole

For the long time from the pioneering paper by Dirac, the most serious argument to support the monopole concept However, as time went on and the idea of grand unification emerged, it seemed that the latter argument had lost some power.

Indeed, the modern point of view is that the operator of electric charge is the generator of a $U(1)$ group. The charge quantization condition arises in models of unification if the electromagnetic subgroup is embedded into a semi-simple non-Abelian gauge group of higher rank. In this case, the electric charge generator forms nontrivial commutation relations with all other gen- erators of the gauge group. Therefore, the electric charge quantization today is considered as an argument in support of the unification approach.

Another essence of unified gauge theories break-through is that while a Dirac monopole could be incorporated in an Abelian theory, some non-Abelian models, like that of Georgi and Glashow [2], inevitably contain monopole-like solutions as it was shown independently by 't Hooft [3] and Polyakov [4].

2.1. $SU(2)$ unified gauge theory

According to 't Hooft [3] and Polyakov [4] let us take a look at non-Abelian $SU(2)$ theory with scalar field in the adjoint representation. The Lagrangian density of such theory is:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{\lambda}{4}(\phi^a\phi^a - v). \tag{2.1}$$

The covariant derivative is defined as:

$$D^\mu\phi^a = \partial_\mu\phi^a + ie[A_\mu, \phi^a].$$

The field strength tensor is

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

where, $a, b = 1, 2, 3$, $A_\mu = ieA^a_\mu t^a$, e and λ are gauge and scalar coupling constants, respectively. The generators of Lie algebra $su(2)$ in fundamental and adjoint representations are $t^a = \sigma^a/2$ and $(T^a)_{bc} = -i\varepsilon_{abc}$, respectively.

After applying expressions for $SU(2)$ notification covariant derivative and strength tensor becomes:

$$D^\mu \phi^a = \partial_\mu \phi^a - eA^b_\mu \varepsilon_{abc}\phi^c \text{ and } F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon_{abc}A^b_\mu A^c_\nu. \tag{2.2}$$

Potential of scalar field ϕ in 2.1 lead to spontaneously broken symmetry $SU(2) \to U(1)$, let us chose the vacuum state of scalar field as $\phi_0 = (0, 0, v)$. Furthermore, let us consider a fluctuation φ of the scalar field ϕ around the trivial vacuum ϕ_0, where only the third isotopic component of the Higgs field is non-vanishing:

$$\phi = (0, 0, v + \varphi). \tag{2.3}$$

Substitution of the expansion 2.3 into the Lagrangian 2.1 yields, up to terms of the second-order:

$$\begin{aligned}\mathcal{L}^{(2)} \approx &- \frac{1}{4}(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu) - \frac{1}{4}(\partial_\mu A^1_\nu - \partial_\nu A^1_\mu) - \frac{1}{4}(\partial_\mu A^2_\nu - \partial_\nu A^2_\mu) \\ &+ \frac{1}{2}e^2v^2((A^1_\mu)^2 + A^1_\nu)^2) + \frac{1}{2}e^2v^2((A^2_\mu)^2 + A^2_\nu)^2) \\ &+ \frac{1}{2}(\partial_\mu\varphi)^2 - 2\lambda v^2\varphi^2.\end{aligned} \tag{2.4}$$

According to 2.4, the perturbative spectrum of theory consists of:

1) massless vector field A^3_μ corresponding to the unbroken electromagnetic subgroup $U(1)$;
2) massive vector fields $W^\pm_\mu = \dfrac{1}{\sqrt{2}}(A^1_\mu \pm iA^2_\mu)$, $m_W = ev$, the electric charge of the massive vector bosons is given by the unbroken $U(1)$ subgroup;
3) massive scalar field φ, $m_\varphi = \sqrt{2\lambda}v$.

After breaking symmetry, the Lagrangian 2.1 becomes:

$$\mathcal{L} \to -\frac{1}{4}(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu),$$

which means, that the Lagrangian satisfy the unbroken electrodynamics $U(1)$.

For the low-energy limit $U(1)$ it is convenient to chose field tensor as [3]

$$F_{\mu\nu} = n^a F^a_{\mu\nu} + \frac{1}{e}\varepsilon_{abc} n^a D_\mu n^b D_\nu n^c, \tag{2.5}$$

here $n^a = \phi^a/\phi$, which for chosen vacuum state becomes $n = (0,0,1)$. After substitution, the field tensor is

$$F_{\mu\nu} = -\frac{1}{4}(\partial_\mu A^3_\nu - \partial_\nu A^3_\mu),$$

which is exactly describe $U(1)$ electromagnetic theory.

2.2. Topological Classification of the Solutions

Beside the perturbative spectrum of Georgi-Glashow theory according to 2.4 there are also stable static solutions of field equations system with finite energy. Let us consider the Langrangian 2.1 in static unitary gauge, which means that $\partial_0 = 0$ and $A_0 = 0$. According to such gauge, the Hamiltonian or the total energy of the system is

$$E = \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{4} F^a_{ij} F^{ij,a} + \frac{1}{2} D_i \phi^a D^i \phi^a + \frac{\lambda}{4}(\phi^a \phi^a - v) \right], \tag{2.6}$$

where $i,j = 1,2,3$. For the finite total energy of the system $E < \infty$, the non-trivial minimum of the energy 2.6 correspond to boundary conditions at $r \to \infty$:

$$\begin{cases} \phi^a \phi^a = v^2, \\ D^i \phi^a = 0, \\ F^a_{ij} = 0. \end{cases} \tag{2.7}$$

The first condition from 2.7 means that the scalar field forms a sphere S^2_ϕ with radius v in isotopic space. One can map the sphere S^2_ϕ onto the boundary space sphere S^2_∞ (with $r \to \infty$):

$$f : S^2_\infty \to S^2_\phi,$$

these maps are characterized by topological numbers $n = \deg f \in \mathbb{Z}$. The topological number correspond to the number of times the S^2_ϕ turn around the S^2_∞. For the map $f : S^2_\infty \to S^2_\phi$, the topological number describes as:

$$n = \frac{1}{8\pi v^3} \int \mathrm{d}\, S_k \varepsilon^{ijk} \varepsilon_{abc} \phi^a \partial_i \phi^b \partial_j \phi^c. \tag{2.8}$$

The trivial case $n = 0$ correspond to asymptotic scalar field $\phi^a = (0, 0, v)$, which means that the orientation of scalar field in isotopic space does not depend on spatial coordinates.

The simplest non-trivial map can be constructed as:

$$\lim_{r\to\infty} \phi^a = v \frac{x^a}{r}, \tag{2.9}$$

which means that isovector of scalar field ϕ^a is directed in the isotopic space as the radius-vector x^a directed in the spatial space at asymptotic $r \to \infty$. Alexander Polyakov in his paper [4] called such solution as «hedgehog» (see fig. 2.1).

The second asymptotic condition in 2.7 with $\phi^a \phi^a = v^2$ describe the asymptotic vector field A^a_i. As it was shown in [3; 5], the solution for $D^i \phi^a = 0$ can be written as:

$$A^a_i = \frac{1}{ev^2} \varepsilon_{abc} \phi^b \partial_i \phi^c + \frac{1}{v} A_i \phi^a, \tag{2.10}$$

where A_i is an arbitrary vector. Substituting 2.10 into expression for strength tensor 2.2 yields to gauge-invariant tensor [6]:

$$F_{ij} = \partial_i A_j - \partial_j A_i + \frac{1}{ev^3} \varepsilon_{abc} \phi^a \partial_i \phi^b \partial_j \phi^c. \tag{2.11}$$

As the strength tensor 2.5 defined by ’t Hooft, the strength tensor 2.11 also describe $U(1)$ electromagnetic theory at low-energy limit, but the differences is that the tensor 2.5 is singular at zero of scalar field (due to normalization), while tensor 2.11 is regular everywhere.

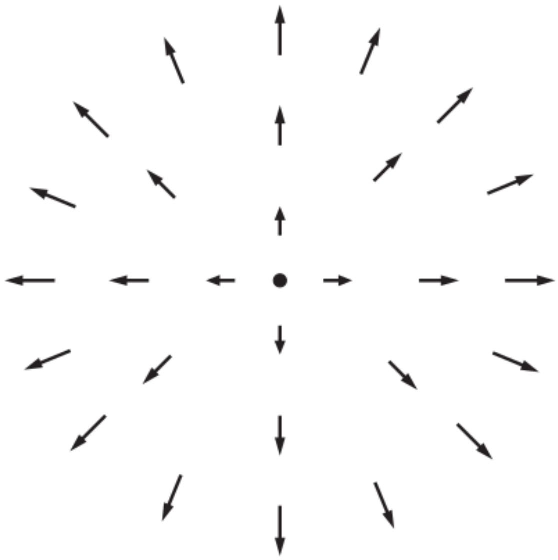

Fig. 2.1 — Configuration of isovector ϕ^a in the isotopic space

With obtained field strength tensor 2.11 one can justify the definition of magnetic charge g as a flux of magnetic field through the infinitely distanced sphere [7]:

$$
\begin{aligned}
g = \int \mathbf{dS}\cdot\mathbf{H} = -\frac{1}{2}\int \mathrm{d}S_i \varepsilon^{ijk} F_{jk} = \\
= -\frac{1}{2}\int \mathrm{d}S_i \varepsilon^{ijk}\left(\partial_j A_k - \partial_k A_j + \frac{1}{ev^3}\varepsilon_{abc}\phi^a \partial_j \phi^b \partial_k \phi^c\right) = \\
= \frac{1}{2ev^3}\int \mathrm{d}S_i \varepsilon^{ijk}\varepsilon_{abc}\phi^a \partial_j \phi^b \partial_k \phi^c = \frac{4\pi n}{e},
\end{aligned}
$$

here were also used $\int \mathrm{d}S_i \varepsilon^{ijk}(\partial_j A_k - \partial_k A_j) = 0$ and expression for topological number n 2.8. Finally, we get charge quantization condition:

$$
g = \frac{4\pi n}{e}. \tag{2.12}
$$

It is very non-trivial result — the non-Abelian theory with spontaneously broken symmetry $SU(2) \to U(1)$ inevitably contain magnetic monopole and furthermore, such monopole is not an elementary particle, according to obtained results, monopole — is a topological defect and the quantization condition depend on topology configuration of monopole!

2.3. Asymptotic properties of 't Hooft–Polyakov monopole

In previous section, incredible results were obtained. The quantization condition 2.12 is similar to Dirac quantization condition 1.18 if we assume $n_D = 2n$, where n_D is Dirac quanting number and n is topological number. And if both monopoles lead to one quantization condition, it is natural to consider the connection between 't Hooft–Polyakov and Dirac monopoles.

As it was shown above, the asymptotic scalar field at $r \to \infty$ can be written as 2.9. According to asymptotic conditions 2.7, vector field becomes 2.10 with the arbitrary vector A_i. Let us assume that $A_i = 0$, and substituting asymptotic scalar field 2.9 into 2.10 yields to asymptotic fields configuration also known as Wu–Yang monopole [8]:

$$\begin{cases} \phi^a = v\dfrac{x^a}{r}, \\ A_i^a = \dfrac{1}{er^2}\varepsilon_{aij}x_j. \end{cases} \tag{2.13}$$

But in the law-energy limit and on the infinitely distanced from monopole region the scalar field get value $\phi^a = (0,0,v)$. Thus, one can define gauge transformation:

$$\begin{cases} \phi = ie\phi^a\dfrac{\sigma^a}{2} \to \omega\phi\omega^{-1}, \\ A_i = ieA_i^a\dfrac{\sigma^a}{2} \to \omega A_i\omega^{-1} + \omega\partial_i\omega^{-1}, \end{cases} \tag{2.14}$$

where σ^a is Pauli matrices (generators $SU(2)$); ω is object of $SU(2)$, which transform asymptotic scalar field inside monopole core 2.13 into vacuum state $\phi^a = (0,0,v)$ correspond to trivial map $n = 0$. The ω could be chosen as:

$$\omega = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{2.15}$$

where θ and ϕ are spherical coordinates. Such gauge transformation is singular in $\theta = \pi$, in such case there is a «sought»-pole singularity due to uncertainty in azimuth angle ϕ.

Applying such singular transformation 2.15 to vector field yields to:

$$\mathbf{A} = \frac{1}{er}\frac{1 - \cos\theta}{\sin\theta}\mathbf{e}_{\phi}, \tag{2.16}$$

which is exactly the vector-potential for Dirac string 1.11 if we considered quantization condition 2.12.

Of course, the gauge transformation between two configurations with different topological numbers cannot be continuously, such transformation always will be singular [7] like 2.15. In case of 't Hooft-Polyakov monopole such gauge transformation leads to the fact that from an infinitely distant perspective such monopole is similar to Dirac monopole with the characteristic Dirac string 2.16. Polyakov called such effect as «comb the hedgehog» — when one try to rotate the isovector ϕ^a in isotopic space to one direction, the Dirac string occur (dashed line on fig. 2.2).

And actually the stability of monopole is secured by the topology — there is no continuous transformations which can deform field configuration of unbroken symmetry inside the monopole to the trivial vacuum state, there will always be an infinity barrier!

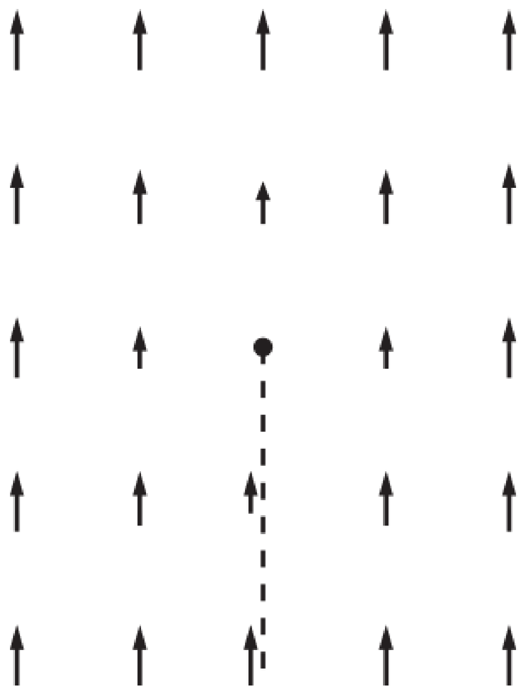

Fig. 2.2 — Noncontinuous gauge transformation of isovector ϕ^a, dashed line correspond to Dirac string (vector-potential)

2.4. Bogomolny-Prosada-Sommerfeld limit

In the previous section it was shown that the field configuration inside monopole asymptotically looks like 2.13. But it is interesting to define the inner structure of the monpole. Thus, one can make use of the 't Hooft–Polyakov ansatz:

$$\begin{cases} \phi^a = \dfrac{x^a}{er^2} H(\xi), \\ A_i^a = \dfrac{1}{er^2} \varepsilon_{aij} x_j (1 - K(\xi)) \end{cases} \tag{2.17}$$

where $H(\xi)$ and $K(\xi)$ are structure functions of dimensionless variable $\xi = ver$. Those structural functions of scalar and gauge fields can be found from field equations, but it is more convenient to use the condition that the monopole itself correspond to the minimum of total energy 2.6. Substituting the ansatz 2.17 into the energy integral leads to:

$$\begin{aligned} E = \frac{4\pi v}{e} \int \frac{d\xi}{\xi^2} \Bigg[\xi^2 \left(\frac{dK}{d\xi} \right)^2 + \frac{1}{2} \left(\xi \frac{dH}{d\xi} - H \right)^2 + \\ + \frac{1}{2} (K^2 - 1)^2 + K^2 H^2 + \frac{\lambda}{4e^2} (H^2 - \xi^2)^2 \Bigg]. \end{aligned} \tag{2.18}$$

Variations of this functional with respect to the functions H and K yields

$$\begin{cases} \xi^2 \dfrac{d^2 K}{d\xi^2} = K H^2 + K(K^2 - 1), \\ \xi^2 \dfrac{d^2 H}{d\xi^2} = 2K^2 H + \dfrac{\lambda}{e^2} H (H^2 - \xi^2). \end{cases} \tag{2.19}$$

The functions H and K must satisfy asymptotic boundary conditions 2.13:

$$\begin{cases} K(\xi) \to 1, \ H(\xi) \to 0 \text{ as } \xi \to 0, \\ K(\xi) \to 0, \ H(\xi) \to \xi \text{ as } \xi \to \infty. \end{cases} \tag{2.20}$$

Unfortunately, the system of non-linear equations 2.19 with boundary conditions 2.20 cannot be solved analytical in general, but there is one very special exception with $\lambda = 0$ called Bogomolny-Prosada-Sommerfeld (BPS) limit [9–11].

The simplest way to get BPS limit is to consider a total energy in general form:

$$E = \int d^3x \left[\frac{1}{2} \big((E_i^a)^2 + (B_i^a)^2 + (D_i \phi^a)^2 \big) + \frac{\lambda}{4} (\phi^a \phi^a - v) \right],$$

with condition $\lambda = 0$ energy can be rewritten as

$$E \geq \frac{1}{2} \int d^3x \big(E_i^a - D_i \phi^a \sin \alpha \big)^2 + \frac{1}{2} \int d^3x \big(B_i^a - D_i \phi^a \cos \alpha \big)^2,$$

here α is arbitrary angle. It is clearly seen, that minimum of the energy corresponds to

$$\begin{cases} E_i^a = D_i \phi^a \sin \alpha, \\ B_i^a = D_i \phi^a \cos \alpha, \end{cases} \tag{2.21}$$

these are the PBS equations. The situation is simple to analyze if the electric charge and electric field vanish, then PBS equations simplifies to

$$B_i^a = D_i \phi^a. \tag{2.22}$$

And returning to the total energy

$$E \geq \int d^3x \big(B_i^a \pm D_i \phi^a \big)^2 \mp B_i^a D_i \phi^a \geq \pm \int B_i^a D_i \phi^a,$$

the subintegral expression can be written as

$$B_i^a D_i \phi^a = \partial_i (B_i \phi^a) - \phi^a D_i B_i^a,$$

according to Bianchi identity, the last part can be vanished $D_i B_i^a = 0$, then:

$$E \geq \pm \int d^3x \partial_i (B_i \phi^a) = \pm \int dS_i H_i^a \phi^a = \pm v \int dS_i H_i = v|g|.$$

After such calculations we finally get Bogomolny bound on monopole mass [9]:

$$M = E \lesssim v|g|. \tag{2.23}$$

Substituting the 't Hooft–Polyakov ansatz 2.17 into simplified BPS equations yields to system:

$$\begin{cases} \xi \dfrac{dK}{d\xi} = -KH, \\ \xi \dfrac{dH}{d\xi} = H + (1 - K^2), \end{cases} \tag{2.24}$$

which have analytical solution:

$$\begin{cases} H(\xi) = \xi \coth \xi - 1 \\ K(\xi) = \dfrac{\xi}{\sinh \xi}. \end{cases} \tag{2.25}$$

Note, that the solution to the first-order BPS equation 2.24 automatically satisfies the system of field equations of the second-order 2.19.

Numerical solutions of the system (5.43) were discussed in the papers [12; 13]. It turns out that the shape functions $H(\xi)$ and $K(\xi)$ approach rather fast to the asymptotic values (see fig. 2.3)

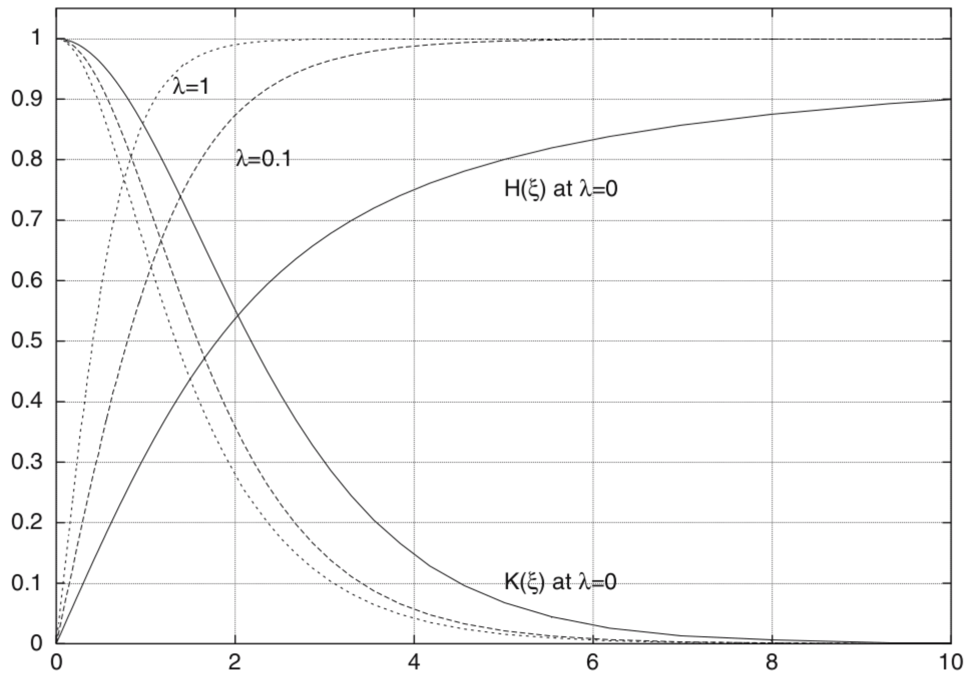

Fig. 2.3 — The functions $K(\xi)$ and $H(\xi)/\xi$ for the 't Hooft– Polyakov monopole at $\lambda = 0$, (BPS limit) $\lambda = 0.1$ and $\lambda = 1$

Also numerical calculations [14] show that the mass of the monopole depends on the scalar coupling constant as

$$M \approx \frac{4nv}{e} f\left(\frac{\lambda}{e^2}\right), \tag{2.26}$$

The smooth function $f(\lambda/e^2)$ is a monotonically increasing function, interpolating between the limits

$$f(0) = 1, \quad f(\infty) = 1.787.$$

The reason why the mass becomes independent of the values of the coupling constant λ for $\lambda \gg 1$ is that in this limit the scalar field approaches the asymptotic form faster than the vector field.

According to numerical results and BPS-solutions, the characteristic mass of monopole is $\sim 1 - 10$ TeV, with the vacuum expectetion value for Georgi-Glashow theory $v \sim 100$ GeV.

3. Magnetic Monopoles and Cosmology

The Particle Physics and the Cosmology are closely related — laws of particle physics are reflected at cosmological processes, especially at early universe stage, when energy density was high and physics beyond Standard Model (SM) may occur.

One of the arguments for SM extension comes from unification of strong and electroweak interactions in symmetries of Grand Unified Theories (GUT). The discovery of nonzero neutrino mass in a form of neutrino oscillations has already moved physics beyond the SM, because in the SM neutrinos stays massless.

If the symmetry is spontaneously broken, it is restored when the temperature exceeds the corresponding scale ($\sim kT$). Such high temperatures should have naturally arisen at the early stages of cosmological evolution. With the cosmological expansion, the temperature decreased, and the phase transition with broken symmetry may reflect at observable cosmological consequences.

It makes Universe a natural laboratory of particle physics, not only due to possibility of the creation of hypothetical stable particles in the early universe, but also owing to the reflection of the broken symmetry in cosmological phase transitions and in their observable effects.

Since 't Hooft-Polyakov monopoles are inevitably produced with the symmetry breaking in Georgi-Glashow model and in any GUT (with characteristic mass $\sim 10^{16}$ GeV), it is interesting to study the effect, which may occur in presence of monopoles at early Universe.

In the following chapter the problem of magnetic monopoles overproduction in Big Bang Theory and its solving in the framework of Inflation theory are considered.

3.1. Magnetic monopoles in Big Bang Theory

In the GUT with compacted $U(1)_{em}$ symmetry magnetic monopoles (and antimonopoles) are predicted with masses exceeding the scale of GTU v_{GUT} (according to 't Hooft and Polyakov):

$$m \sim \frac{v_{GUT}}{e} \sim 10^{16} \text{ GeV}, \tag{3.1}$$

which makes it impossible to search for such monopoles on colliders.

Moreover, no artificial or natural source of particles in the modern Universe can provide the conditions for the production of particles with such gigantic masses. But such energy scale may be reached at early universe, which may lead to natural production of magnetic monopoles.

According to old Big Bang Theory, particles with mass m is in thermodynamic equilibrium if:

$$T > m, \tag{3.2}$$

at this condition, the reaction rate is

$$n(T)\langle\sigma v\rangle > H, \tag{3.3}$$

where $n(T)$ — concentration of particles (antiparticles) at temperature T, σ — reaction cross-section, v — velocity of particles, $H \sim T^2/M_{Pl}$ — cosmological expansion rate (Hubble parameter).

When the temperature decrease at level $T < m$, but condition 3.3 is satisfied, the concentration reach its equilibrium value:

$$n(T) = \left(\frac{2}{\pi^3}\right)(mT)^{1/3}\exp\left[-\frac{m}{T}\right]. \tag{3.4}$$

At temperature T_f, when reaction rate reach cosmological expansion rate H, the particles are out of equilibrium, their relative concentrations are not constant anymore – it is called freez-out.

Based on such process, one can estimate primordial concentration of magnetic monopoles, consider that at high temperature monopole-antimonopoles pairs were in equilibrium.

Following Zeldovich and Khlopov [15], consider annihilation reaction of monopole-antimonopole in the early Universe. Annihilation reaction is defined by Coulomb attraction of magnetic charges g. At the temperature T monopole-antimonopole Coulomb attraction of magnetic charges is essential at distances:

$$r \leq r_0 = \frac{g^2}{T}. \tag{3.5}$$

In the case when the free pass in a plasma $\lambda \gg r_0$, free annihilation reaction can be considered, in the opposite case, $\lambda \ll r_0$ annihilation first should be treated in the diffusion approximation, and only then as a free annihilation [15].

Since the concentration of monopoles in several order of magnitude lesser than concentration of relativistic particles n_γ, the free pass is determined as:

$$\lambda = \frac{1}{\langle n_{ch}\sigma\rangle}, \tag{3.6}$$

According to [16], The cross-section of monopole multiple scattering at $90°$ is given as:

$$\sigma \sim \frac{(ge)^2}{Tm}, \tag{3.7}$$

after which, the diffusion approximation is applicable up to time

$$t_1 \lesssim \frac{m_{Pl}}{\alpha^2 m^2} \tag{3.8}$$

which is corresponded to the temperature $T \lesssim eT_{cr}$.

To find annihilation rate, let us consider the diffusion equation of monopoles and antimonopoles in an absorbing sphere with radius $a \leq r_0$:

$$\frac{\partial n(r,t)}{\partial t} = \frac{D}{r^2} \cdot \frac{\partial}{\partial r} r^2 \left(\frac{\partial n(r,t)}{\partial r} + \frac{g^2}{Tr^2} n(r,t) \right), \tag{3.9}$$

where $D = 1/3 \cdot \lambda v$ is a diffusion coefficient. In static case $\partial n/\partial t = 0$, with boundary conditions:

$$n(\infty) = n_m, \quad n(a) = 0,$$

the solution of the equation 3.9 is:

$$n(r) = \begin{cases} 0, & r \leq a, \\ n_0 \dfrac{1 - \exp(r_0/r - r_0/r)}{1 - \exp(-r_0/a)}, & r > a. \end{cases} \tag{3.10}$$

Then the diffusion flux is given by:

$$\Phi = 4\pi r^2 D \frac{\partial n(r)}{\partial r} \approx 4\pi D r_0 n_m, \tag{3.11}$$

and the annihilation rate is:

$$\frac{dn_m}{dt} = -n_m^2 4\pi D r_0. \tag{3.12}$$

Now the equation for the relative monopole concentration $\nu = n_m/n_\gamma$ with account of equation 3.12 in case of monopoles going out of thermodynamic equilibrium (up to time t_1 3.8) is given by [17; 18]:

$$\frac{d\nu}{dt} = -4\pi D r_0 n_\gamma \nu^2 = -A\theta^{1/2}\nu^2, \tag{3.13}$$

where $\theta = T/m$ is a dimensionless temperature and

$$A = \frac{4\pi}{3}\frac{g}{e}m.$$

Solution for equation 3.13 is

$$\nu(t) = \frac{\nu(t_0)}{1 + \dfrac{4}{3}A\theta_0^{1/2}t_0(\tau^{3/4} - 1)\nu(t_0)}, \tag{3.14}$$

where $\tau = t/t_1$.

According to [15], further annihilation of free monopoles does not change results of diffusion approximation and concentration of relic monopoles can be taken as:

$$\nu(\infty) \approx \nu(t_1).$$

Thus, if primordial monopole's concentration is

$$\nu_0 \ll \frac{3}{4A\theta_1^1/2} t_1, \tag{3.15}$$

then concentration of relic monopoles does not depend on ν_0 and density of relic monopoles with mass $m \sim 10^{16}$ GeV can be written as:

$$\Omega_m = m \frac{m}{g^5(eg)m_{Pl}} \frac{n_\gamma}{\epsilon_{cr}} \sim 10^{15} \left(\frac{v_{GUT}}{10^{16}\ \mathrm{GeV}} \right), \tag{3.16}$$

which is at least on 15 orders of magnitude grater than barionic density!

In fact, diffusion slow down annihilation rate with respect to the free annihilation, which means that more monopoles should survive.

3.2. Inflation as a solution of monopole overproduction problem

text text text

4. Searches for ultraheavy magnetic monopoles

In this chapter, briefly discussion on experiments for searching of ultraheavy magnetic monopoles is presented.

Particles carrying magnetic charge, similarly to electric charge, will deposit some amount of energy through the ionization process and excitation of atoms when they traverse matter. To calculate the energy loss of monopoles passing through matter, the energy transfer of the monopole to the surrounding medium is considered.

There are three primary ways via which the energy can be dissipated and their importance depends on the monopole velocity and the medium: ionization, atomic excitation and elastic collisions with atoms.

Modern experiments such as IceCube, NOvA, Macro search for monopoles in width range of its velocity from $\beta \sim 10^{-4}$ up to $\beta \sim 1$.

Magnetic monopoles can gain kinetic energy through acceleration in magnetic fields. This acceleration follows from a generalized Lorentz force law [19]. The kinetic energy gained by a monopole of charge g traversing a magnetic field B with coherence length L is about $E \sim gBL$ [20]. At such high kinetic energies magnetic monopoles can pass through the Earth while still having relativistic velocities.

The magnetic charge g moving with a velocity $\beta = v/c$ produces an electrical field whose strength is proportional to the particle's velocity and charge. At velocities above $v_c = c/n_p$, where n_p is the refraction index of matter, Cherenkov light is produced analogous to the production by electrical charges in an angle θ of

$$\cos\theta = \frac{1}{n_p\beta}. \tag{4.1}$$

The number of Cherenkov photons per unit path length x and wavelength λ emitted by a monopole with one magnetic charge g can be described by the usual Frank-Tamm formula for a particle with effective charge $Ze \to gn_p$ [21]:

$$\frac{d^2N_\gamma}{dxd\lambda} = \frac{2\pi\alpha}{\lambda^2}\left(\frac{gn_p}{e}\right)^2\left(1 - \frac{1}{\beta^2 n_p^2}\right). \tag{4.2}$$

For example, for IceCube detector, a minimal charged monopole generates 8200 times more Cherenkov radiation in ice (with $n_p \approx 1.32$) than an electrically charged particle with the same velocity (see fig. 4.1) [22].

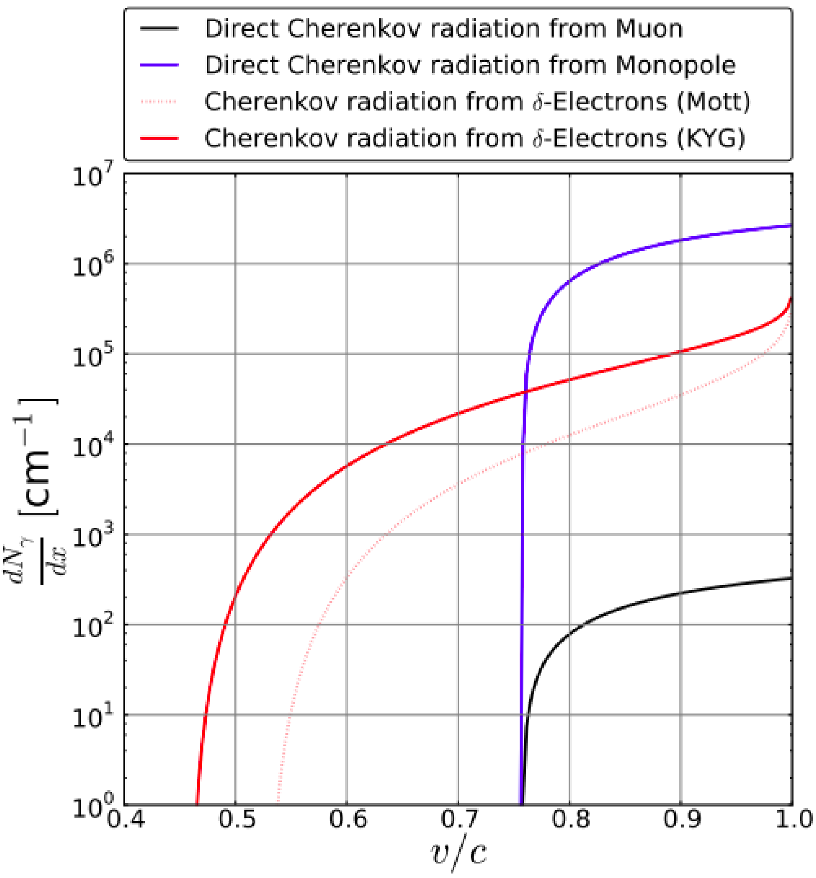

Fig. 4.1 — Number of photons per cm produced by different sources

In addition to this effect, a (mildly) relativistic monopole knocks electrons off their binding with an atom. These high energy -electrons can reach velocities above the Cherenkov threshold v_c. For the production of -electrons the Kasama-Yang-Goldhaber (KYG) cross-section is used to calculate the energy transfer from the monopole to the -electron, and as a result inderect Cherenkov radiation is produced.

With account the KYG cross-section, the energy loss of monopole can be calculated as:

$$\frac{dE}{dx} = \frac{4\pi N_e g^2 e^2}{m_e c^2}\left(\ln\frac{2m_e c^2\beta^2\gamma^2}{I} + \frac{K(g)}{2} - \frac{\delta+1}{2} - B(g)\right), \tag{4.3}$$

where N_e is electron density, γ is Lorentz factor of monopole, I is the ionization potential, $K(g)$ is QED correction for the KYG cross-section, δ is the density effect correction [23] and $B(g)$ is a Bloch correction.

Thus, the brightness of the Cherenkov detector is

$$\frac{dN_\gamma}{dx} = \frac{dN_\gamma}{dE}\frac{dE}{dx}, \tag{4.4}$$

with energy loss 4.3 and detector luminescence light dN_γ/dE.

The background of a monopole search consists of all other known particles which are detectable. The most abundant background are muons or muon bundles produced in air showers caused by cosmic rays. The majority of neutrino induced events are caused by neutrinos created in the atmosphere. Conventional atmospheric neutrinos, produced by the decay of charged pions and kaons, are dominating the neutrino rate from the GeV to the TeV range. Prompt neutrinos, which originate from the decay of heavier mesons, i.e. containing a charm quark, are strongly suppressed at these energies.

On the fig. 4.2 shown limits for relativistic ultraheavy monopoles from experiments BAIKAL [24], ANTARES [25], IceCube [22; 26] and up-going results from MACRO [27].

Dashed line on the fig. 4.2 correspond to Parker limit [28]

$$\Phi \lesssim 10^{-15}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}\mathrm{sr}^{-1},$$

which is caused by primordial magnetic field.

Energy loss for slow monopoles differ from 4.3 and, according to Ahlen and Kinoshita [29], it can be taken as:

$$\frac{dE}{dx} = aN_e^{2/3}\left[\ln\left(bN_e^{1/3}\right) - cfrac12\right]\beta, \tag{4.5}$$

where the constants a and b do not depend on material and are defined as:

$$a = \frac{2\pi g^2 e^2}{(3\pi^2)^{1/3}}, \quad b = 2(3\pi^2)^{1/3} a_0, \tag{4.6}$$

a_0 is the Bohr radius. Note that 4.5 is for non-catalyzing slow monopoles, and it is applicable at $\beta < 10^{-2}$. On the fig. 4.3 shown experimental limits on non-relativistic monopoles obtained with NOvA [30], going-up MACRO [27] experiments.

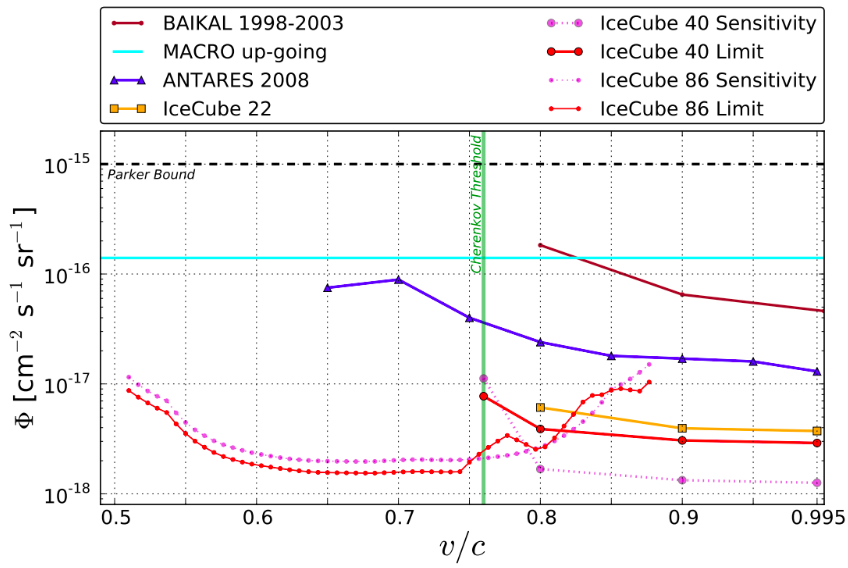

Fig. 4.2 — Experimental limits on relativistic ultraheavy monopoles flux

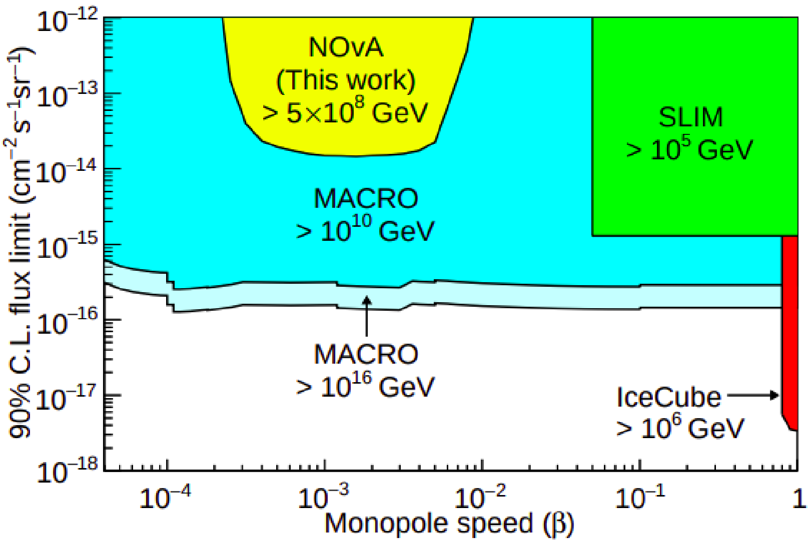

Fig. 4.3 — Experimental limits on non-relativistic ultraheavy monopoles flux

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