Thermodynamics of hot Universe

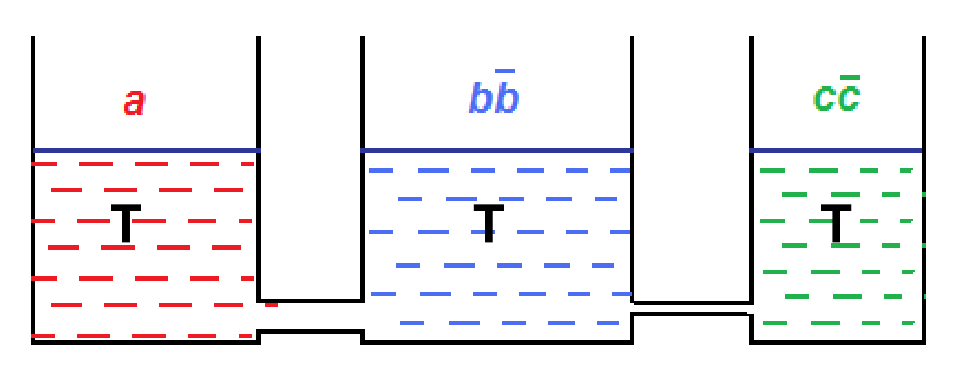
Lecture from the course

« Introduction to cosmoparticle physics »

Equilibrium condition

$$\Gamma_{ab} = n_{ab}\sigma_{ab}v_{ab} >> \Gamma_{\substack{\text{macroscopic}\\ \text{conditions}}}$$

- condition of equilibrium between species *a* and *b*.



For matter in Universe, the change of macroscopic parameters is defined by the rate of its expansion:

$$\Gamma_{\substack{\text{macroscopic}\\ \text{conditions}}} = H \sim \frac{1}{t}$$

Equilibrium distribution

Under conditions of equilibrium, for gases of fermions and bosons we have

$$f = \frac{d^6 N}{d^3 x d^3 p} = \frac{1}{(2\pi\hbar)^3} \frac{g_S}{\exp\left(\frac{E-\mu}{T}\right) \pm 1}$$

$\hbar = c = k = 1$

$\mu \to 0$ chemical potential is supposed to be 0: number of any species can be freely changed

Using this distribution, we can find number and energy densities

$$n = \frac{d^3 N}{d^3 x} = \int f d^3 p \qquad \varepsilon = \int E \cdot f d^3 p$$

Number and energy densities

Ultrarelativistic case: $E=p$, $d^3p=4\pi E^2 dE$

$$n = \frac{g_S}{(2\pi)^3}\int_0^\infty \frac{4\pi E^2 dE}{\exp\left(\frac{E}{T}\right)\pm 1} = \left(x \equiv \frac{E}{T}\right) = \frac{g_S T^3}{2\pi^2}\int_0^\infty \frac{x^2 dx}{e^x \pm 1} = \frac{g_S T^3}{2\pi^2} I_2^{(f/b)}$$

$$\varepsilon = \int E\cdot f d^3 p = \ldots = \frac{g_S T^4}{2\pi^2}\int_0^\infty \frac{x^3 dx}{e^x \pm 1} = \frac{g_S T^4}{2\pi^2} I_3^{(f/b)}$$

Notation is introduced: $I_n^{(f/b)} \equiv \int_0^\infty \frac{x^n dx}{e^x \pm 1}$

The given integrals are not trivial, however relation between them can be calculated

$$I_n^{(b)} - I_n^{(f)} \equiv \int_0^\infty \frac{x^n dx}{e^x - 1} - \int_0^\infty \frac{x^n dx}{e^x + 1} = \int_0^\infty \frac{2x^n dx}{e^{2x} - 1} = \left(y = 2x\right) = \frac{1}{2^n}\int_0^\infty \frac{y^n dy}{e^y - 1} = \frac{1}{2^n} I_n^{(b)}$$

$$\boxed{I_n^{(f)} = \left(1 - \frac{1}{2^n}\right) I_n^{(b)}}$$

Relativistic particles

From formula above we get in ultrarelativistic case

$$n_f = \frac{3}{4} n_b \qquad \varepsilon_f = \frac{7}{8} \varepsilon_b$$

Full calculation gives

$$n = \begin{cases} \dfrac{\zeta(3) g_S T^3}{\pi^2} & \text{- for bosons -} \\ \dfrac{3\zeta(3) g_S T^3}{4\pi^2} & \text{- for fermions -} \end{cases} \qquad \varepsilon = \begin{cases} \dfrac{\pi^2 g_S T^4}{30} \\ \dfrac{7\pi^2 g_S T^4}{240} \end{cases}$$

$\zeta(3) = 1.202$

Compare with Stefan-Boltzman law

$u = \varepsilon_\gamma / 4 = \sigma T^4$

Nonrelativistic particles

In non-relativistic case we have: $E \approx m >> T$

$$n = \frac{g_S}{(2\pi)^3}\int \frac{4\pi p^2 dp}{\exp\left(\frac{m+E_{kin}}{T}\right)\pm 1} \approx \left(\begin{array}{c} e^{m/T} >> 1 \\ p = mv \\ E_{kin} = \frac{mv^2}{2} \end{array} \right) \approx$$

$$\approx \frac{g_S m^3}{2\pi^2}\exp\left(-\frac{m}{T}\right)\cdot \underbrace{\int_0^\infty \exp\left(-\frac{mv^2}{2T}\right) v^2 dv}_{\sqrt{\frac{\pi}{2}}\left(\frac{T}{m}\right)^{3/2}} = g_S\left(\frac{Tm}{2\pi}\right)^{3/2}\exp\left(-\frac{m}{T}\right)$$

Thus

$$\boxed{n = g_S\left(\frac{mT}{2\pi}\right)^{3/2}\exp\left(-\frac{m}{T}\right)}$$

both for bosons and fermions

$$\boxed{\varepsilon = mn}$$

Multicomonent relativistic gas

$$\varepsilon_{tot} = \sum_b \varepsilon_b + \sum_f \varepsilon_f = \left(\sum_b g_{S(b)} + \frac{7}{8} \sum_f g_{S(f)} \right) \frac{\pi^2}{30} T^4 =$$

$$= \underbrace{\left(1 + \sum_{b \neq \gamma} \frac{g_{S(b)}}{2} + \frac{7}{8} \sum_f \frac{g_{S(f)}}{2} \right)}_{\kappa_\varepsilon} \bar{\sigma} T^4 \qquad \bar{\sigma} = 4\sigma$$

$$\varepsilon^{(\text{eq})}_{\text{non-rel}} << \varepsilon^{(\text{eq})}_{\text{rel}}$$

In case of components with different temperatures:

$$\kappa_\varepsilon = 1 + \sum_{b \neq \gamma} \frac{g_{S(b)}}{2} \left(\frac{T_b}{T} \right)^4 + \frac{7}{8} \sum_f \frac{g_{S(f)}}{2} \left(\frac{T_f}{T} \right)^4$$

Equation of state

The basic equations of state, as mentioned previously, are

$p = 0$ -non-relativistic ("dust"-like) matter, the stage of dominance of such matter is called **MD-stage**

$p = \frac{\varepsilon}{3}$ - (ultra)relativistic (radiation-like) matter, the corresponding stage is called **RD-stage**

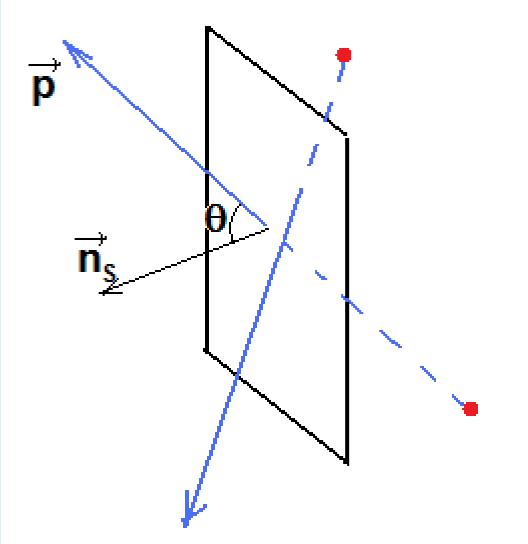
$p = -\varepsilon$ - vacuum-like matter (vacuum energy), this stage leads to accelerated expansion **(inflation)**

In the general case one can parameterize

$$p = \gamma\varepsilon$$

Derivation of p=ε/3

For the pressure of gas of photons we have



$$p \equiv \frac{F_\perp}{S} = \frac{\left\langle \vec{n}_S d\vec{P}/dt \right\rangle}{S} = \frac{\left\langle \vec{n}_S \cdot \vec{p} dN_\gamma / dt \right\rangle}{S}$$

$$dN_\gamma = n_\gamma \cdot S \cdot cdt \cdot \cos\theta$$

$$\vec{n}_S \vec{p} = \left|\vec{p}\right| \cos\theta = E/c \cos\theta$$

$$\Rightarrow \quad p = \left\langle E \cdot n_\gamma \cdot \cos^2\theta \right\rangle = \left\langle E \cdot n_\gamma \right\rangle \left\langle \cos^2\theta \right\rangle = \varepsilon \cdot \frac{1}{3}$$

$$\boxed{p = \frac{\varepsilon}{3}}$$

Basic relations

For the matter with equation of state $p=\gamma\varepsilon$, we can get from Friedmann equations

$$\varepsilon \propto a^{-3(1+\gamma)} \propto \begin{cases} a^{-4} & \text{for relativistic matter} \\ a^{-3} & \text{for non-relativistic matter} \\ const & \text{for vacuum energy.} \end{cases}$$

In early Universe density of CMB exceeded the density of matter.

=> Radiation Dominated (RD)-stage took place at T>1 eV.

For $\gamma \neq -1$

$$\varepsilon_{crit} = \frac{1}{6(1+\gamma)^2 \pi G} \frac{1}{t^2}$$

$$H = \frac{2}{3(1+\gamma)t}$$

$$a \propto t^{\frac{2}{3(1+\gamma)}} = \begin{cases} t^{1/2} & \text{- for RD} \\ t^{2/3} & \text{- for MD} \end{cases}$$

Note, that given relations take place for flat Universe (K=0 or Ω=1) without Λ-term. Such approximation is justified, since the terms K/a^2 and, moreover, $2\Lambda/3$ in Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\varepsilon}{3} = -\frac{K}{a^2} + \frac{2\Lambda}{3}$$

become negligible while a decreases even if $K,\Lambda \neq 0$.

Vacuum dominance

In case of $\gamma=-1$ we have

$$\varepsilon = -p = \frac{\Lambda}{8\pi G}$$

$$H = \sqrt{\frac{8\pi G\varepsilon}{3}}$$

$$a \propto \exp(Ht)$$

Λ-term is equivalent to the matter with e.s. $p=-\varepsilon$ (vacuum energy).

Density of Λ does not change with time.

=> Then Λ-dominance can start only in a late period, provided that small Λ exists.

Task: For homogeneous massive scalar field from general expression of energy-momentum tensor please show that it leads to vacuum equation of state at $t \ll 1/m$.

Temperature of early Universe

Since wavelength of free particle ~a, temperature of photons evolves as a^{-1}.

$$T \propto \frac{1}{\lambda} \propto \frac{1}{a} \propto z+1$$

However, before recombination (T>3000 K, z>1100) and, in particular, at RD stage, photons are not free and can get/give the energy from/to other matter with which they interact (are in equilibrium).

To define dependence of T from t at RD-stage, one writes

$$\varepsilon_{crit}(\gamma = 1/3) = \frac{3}{32\pi G}\frac{1}{t^2} = \kappa_\varepsilon \bar{\sigma} T^4$$

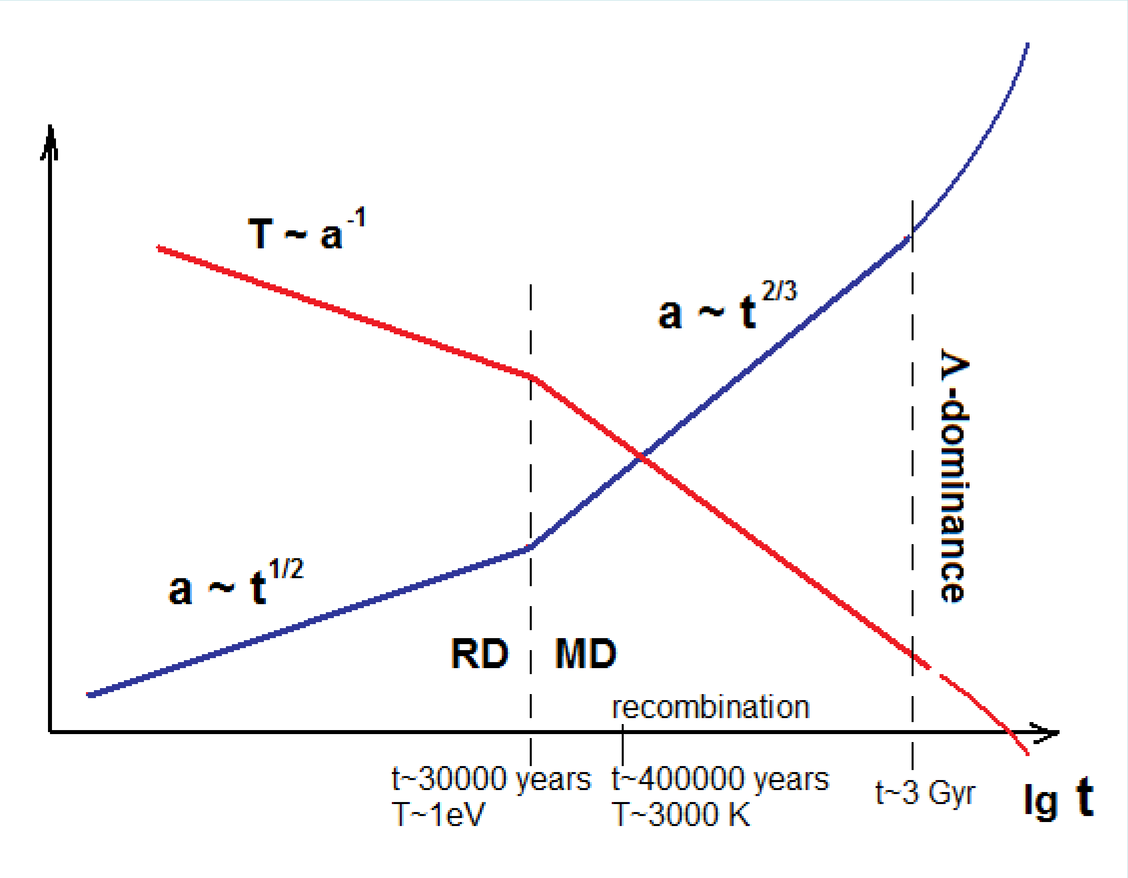
$$T = \left(\frac{45}{32\pi^3 G}\right)^{1/4} \frac{1}{\kappa_\varepsilon^{1/4} t^{1/2}} \approx \left(\text{for } T \sim 1\,\text{MeV}\right) \approx 0.86\,\text{МэВ} \cdot \sqrt{\frac{1\,\text{c}}{t}}$$

Contribution of species

$$\kappa_\varepsilon(T \sim 1\,\text{МэВ}) = 1 + \frac{7}{8}\left(2 \cdot \frac{2}{2}(e^{\pm}) + 3 \cdot 2 \cdot \frac{1}{2}(\nu\bar{\nu})\right) = \frac{43}{8}$$

- κ_ε depends on *t* (*T*) as soon as the number of relativistic species changes with *T*.
- Contribution of non-relativistic species at RD-stage is suppressed as $\exp(-m/T)$ or defined, as in case of nucleons, by small initial excess of their particles over antiparticles.

Evolution with time



Entropy

$$S = \int \frac{dQ}{T}$$

characterizes amount of states in phase space occupied by system.

$$s = \frac{S}{V} = \frac{\varepsilon + p}{T} = \left(p = \frac{\varepsilon}{3} \right) = \frac{4\varepsilon}{3T}$$

Gravitational energy is not taken into account

$$s \propto n$$

Entropy is conserved for reversible processes.

Entropy is conserved for any closed (sub)system in the absence of irreversible processes.

Examples of irreversible processes: radiation of hot bodies (stars), decays of particles, some phase transitions.

Entropy of multicomponent matter

For **multicomponent** matter we have:

$$s_{tot} = \sum_b s_b + \sum_f s_f = \frac{4}{3T}\varepsilon_{tot} = \underbrace{\left(1 + \sum_{b\neq\gamma} \frac{g_{S(b)}}{2} + \frac{7}{8}\sum_f \frac{g_{S(f)}}{2}\right)}_{\kappa_S}\frac{4}{3}\bar{\sigma}T^3$$

κ_S $= \kappa_\varepsilon$ in case of $T_i = T$

In case of components with different temperatures:

$$\kappa_s = 1 + \sum_{b\neq\gamma} \frac{g_{S(b)}}{2}\left(\frac{T_b}{T}\right)^3 + \frac{7}{8}\sum_f \frac{g_{S(f)}}{2}\left(\frac{T_f}{T}\right)^3$$

Freezing out and decoupling

Freezing out of particles *a* (and their antiparticles) takes place, when they go out of **thermodynamic** equilibrium with particles *b*. It happens when processes that maintain the equilibrium, including reactions changing number of particles *a*, are stopped ("frozen out") – become slower than the rate of cosmological expansion (*H*).

$$n_a \sigma_{a\bar{a}} v_{a\bar{a}} = H$$ or $$n_a \sigma_{ab} v_{ab} = H$$

Decoupling of particles *a* from particles *b* takes place, when they go out of **thermal** (kinetic) equilibrium. It happens when energy exchange between *a* and *b*, carried out by their scattering processes, becomes ineffective – becomes slower than Universe expansion.

$$n_a \sigma_{ab} \frac{\Delta E_{ab}}{E_a} v_{ab} = H$$ It takes the form $$n_a \sigma_{ab} v_{ab} = H$$, if $\Delta E_{ab} \sim E_a$

These notions play important role in particle physics of Big Bang Universe

Conclusions

- Rate of processes between matter and radiation exceeds the rate of expansion in early Universe.
- Expansion of Universe reproduces an adiabatic process (adiabatic cooling and conservation of entropy).
- It proves the validity of thermodynamical description for particles in early Universe.
- The conditions of equilibrium, freezing out or decoupling are important for evolution of particles at early hot stages of cosmological evolution.