General relativistic weak-field limit and Newtonian N-body simulations

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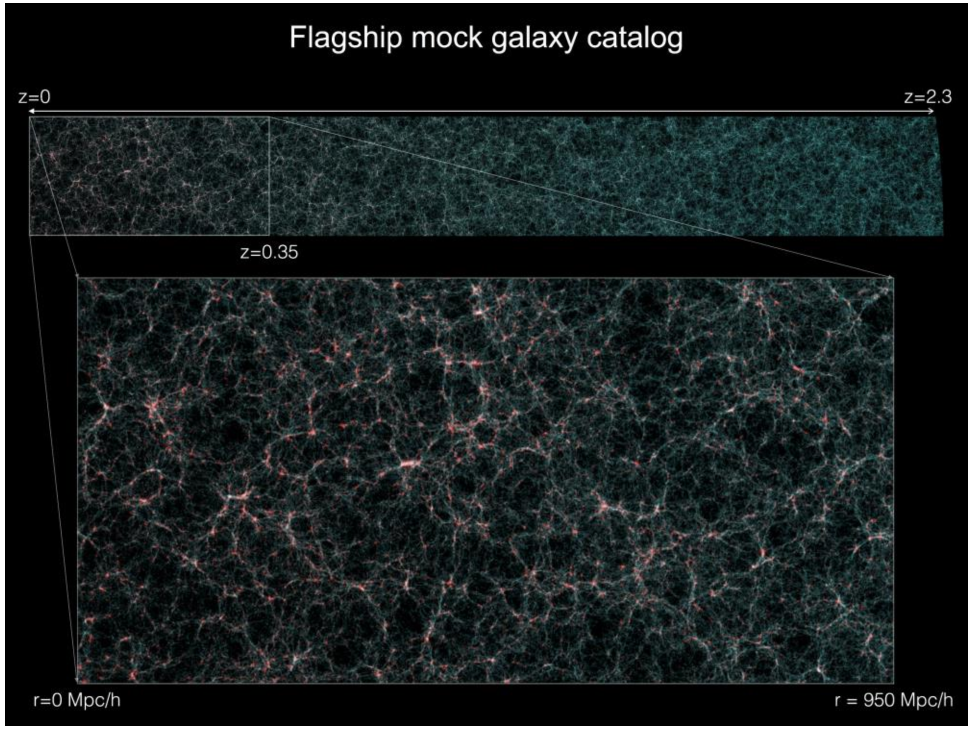
Motivation

- Future surveys (DESI, LSST, Euclid, SKA …)

  These surveys will go wider and deeper,
  probing near horizon perturbations

- N-body simulations

  These surveys require large volume simulations
  cf. Euclid flagship simulation L=3.8 Gpc, N=12600^3
  mock galaxies up to z=2.3



- Limitations of Newtonian simulations

  Newtonian dynamics is based on “action-at-a-distance” in absolute space and time

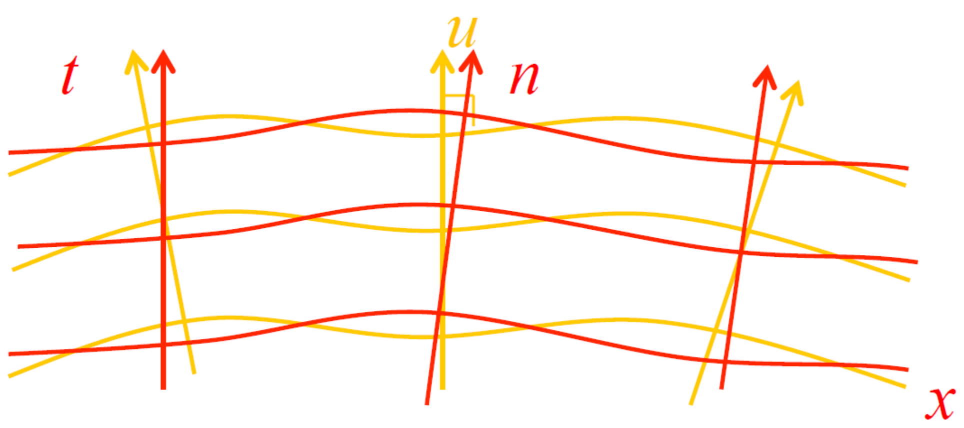
Questions

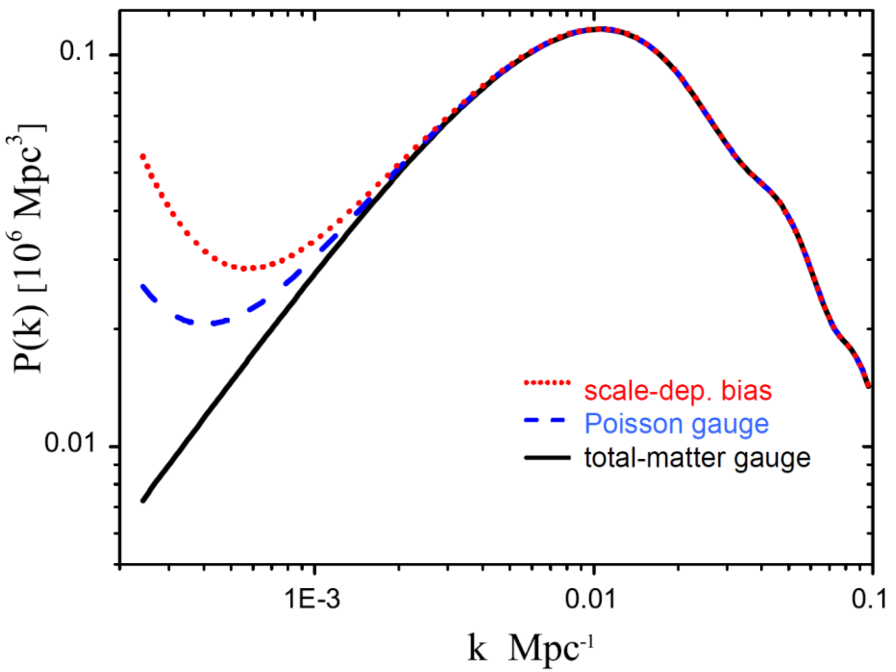
- Are Newtonian N-body simulations consistent with weak-field limit of GR?
- If so, how do we interpret Newtonian simulations in a relativistic framework?
- How do we include relativistic effects missing in simulations (e.g. radiation perturbations)

Gauge

No unique choice of time (slicing) and space coordinates (threading) in an inhomogeneous spacetime

$$t \to t + T(t, x^i) \qquad \rho(t) \to \rho\Big(t + T(t, x^i)\Big) = \rho(t) + \rho'(t) T(t, x^i)$$





Newtonian simulations

- Initial conditions

$$\boldsymbol{x}(\boldsymbol{q},\eta) = \boldsymbol{q} + \boldsymbol{\psi}(\boldsymbol{q},\eta) \qquad -\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_\alpha = \delta_\alpha$$

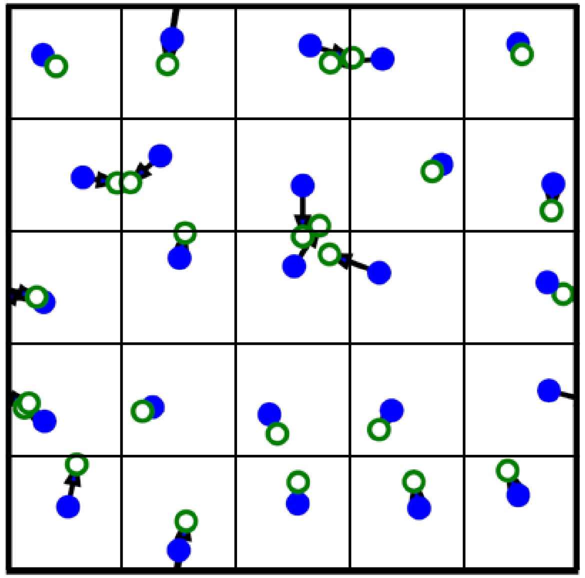
- N-body simulations

$$\rho_{\rm count} = \frac{1}{a^3}\sum_{\rm particles} m\,\delta_{\rm D}^{(3)}(\boldsymbol{x}-\boldsymbol{x}_p)$$

$$k^2\Phi^{\rm N} = 4\pi G a^2 \bar{\rho}_{\rm cdm}\delta^{\rm N}_{\rm cdm}$$

$$\ddot{\boldsymbol{x}}_i = -\nabla\Phi^{\rm N}$$

Newtonian evolution



linearisation

$$k^2\Phi^{\rm N} = 4\pi G a^2 \bar{\rho}^{\rm N}_{\rm cdm}\delta^{\rm N}_{\rm cdm}$$

$$\dot{\delta}^{\rm N}_{\rm cdm} + k v^{\rm N}_{\rm cdm} = 0\,,$$

$$\left[\partial_\tau + \mathcal{H}\right] v^{\rm N}_{\rm cdm} = -k\Phi^{\rm N}\,,$$

N-body gauge (linear perturbations) Fidler et.al. arXiv:1505.04756

- N-body gauge

$$g_{00} = -a^2(1+2A)\,,$$
$$g_{0i} = -a^2 B_i\,,$$
$$g_{ij} = a^2\left[\delta_{ij}\left(1+2H_{\rm L}\right) - 2H_{{\rm T}\,ij}\right]$$

$$B^{\rm Nb} = v^{\rm Nb} \quad H_{\rm L}^{\rm Nb} = 0$$

$$\dot\delta^{\rm Nb}_{\rm cdm} + k v^{\rm Nb}_{\rm cdm} = 0\,,$$
$$\left[\partial_\tau + \mathcal{H}\right] v^{\rm Nb}_{\rm cdm} = -k\left(\Phi + \gamma^{\rm Nb}\right)$$
$$\Phi \equiv H_{\rm L} + \frac{1}{3}H_{\rm T} + \mathcal{H}k^{-1}\left(B - k^{-1}\dot H_{\rm T}\right)$$
$$-k^2\gamma \equiv \left(\partial_\tau + \mathcal{H}\right)\dot H_{\rm T} - 8\pi G a^2 \bar p \Pi \quad \text{(anisotropic stress)}$$

Cold Dark Matter (CDM) + C.C. $\quad H_{\rm T}^{\rm Nb} = 3\zeta = \text{constant} \quad \Pi = 0 \;\Rightarrow\; \gamma$ vanishes

- Relativistic density

$$\zeta = H_{\rm L} + \frac{1}{3}H_{\rm T} + \mathcal{H}k^{-1}(B - v)$$

$$\rho = (1 - 3H_{\rm L})\rho_{\rm count} \qquad \nabla^2\Phi = -4\pi G a^2 \sum \bar\rho_\alpha \delta^{\rm Nb}_\alpha$$

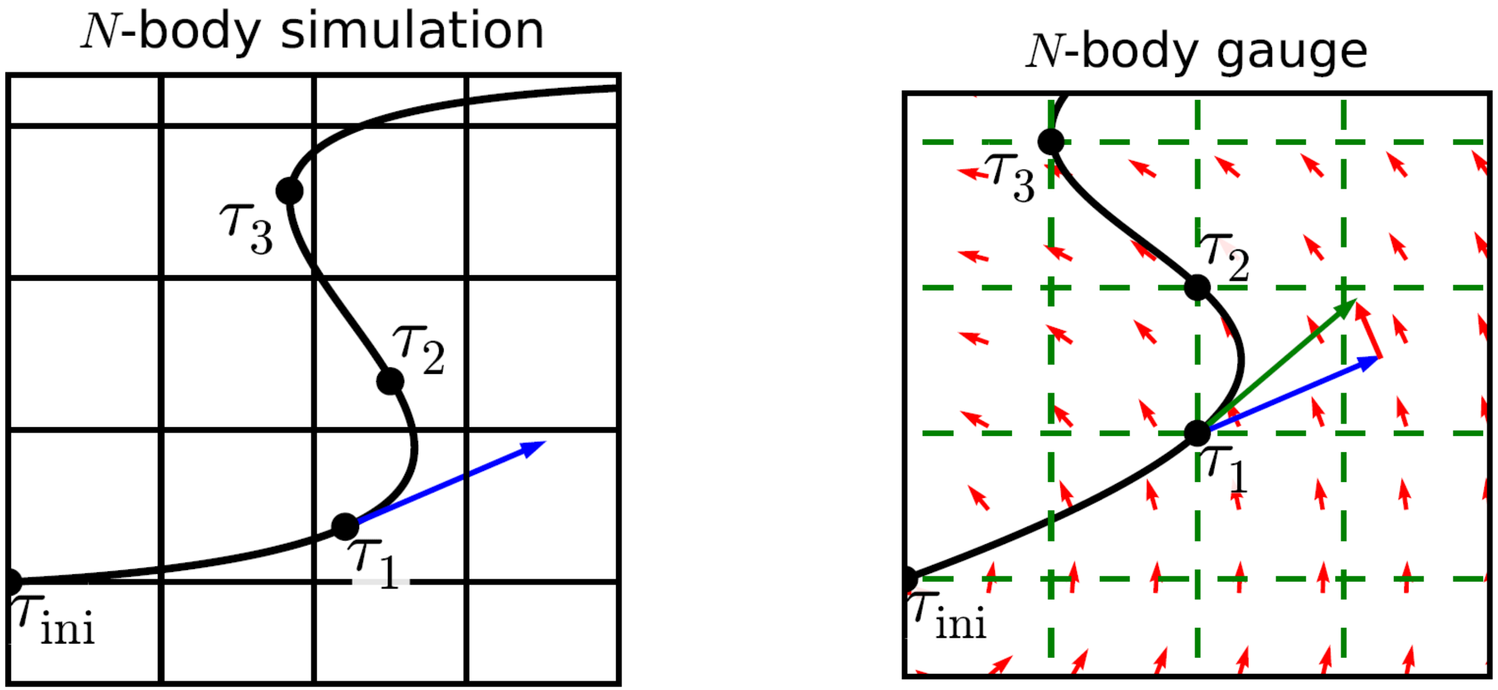
Traceless part of 3-metric does not distort volume in the N-body gauge

Radiation perturbations

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Radiation perturbations

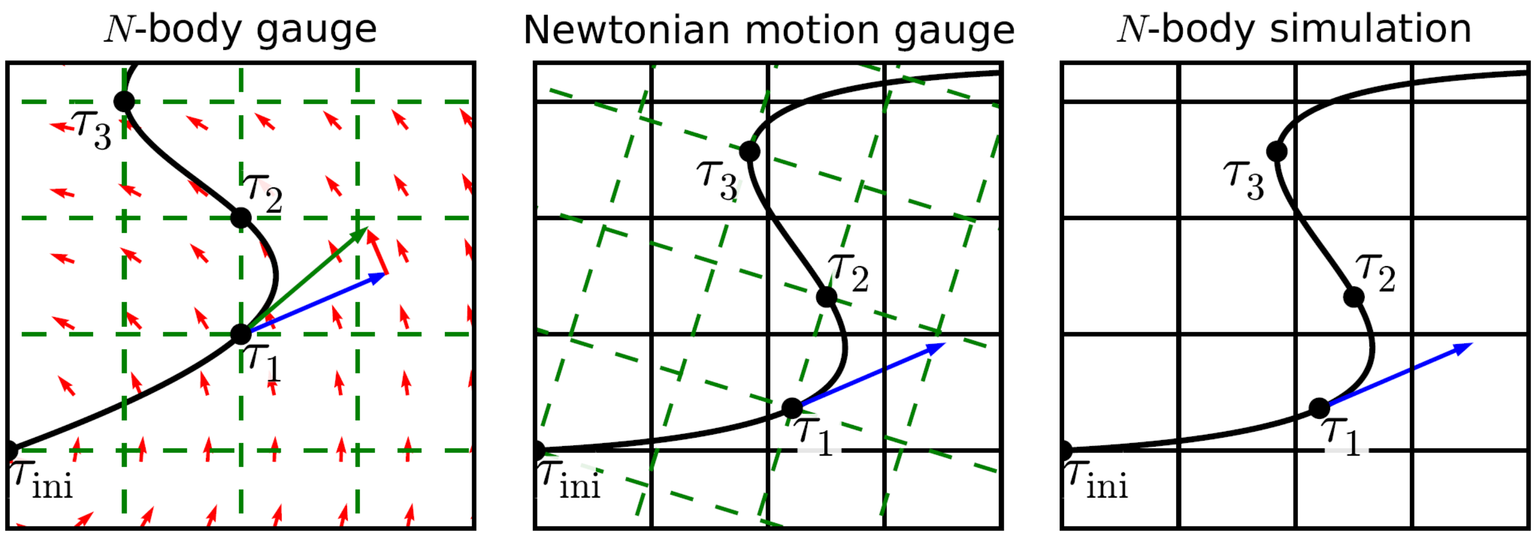
$$[\partial_\tau + \mathcal{H}]\, \boldsymbol{v}^{\mathrm{Nb}}_{\mathrm{cdm}} = \nabla\Phi + \nabla\gamma^{\mathrm{Nb}}$$



Newtonian motion gauge

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Newtonian motion gauge



$$k^2\Phi^{\rm N} = 4\pi G a^2 \bar{\rho}_{\rm cdm}\delta^{\rm N}_{\rm cdm}$$

$$\delta^{\rm N}_{\rm cdm} \equiv \delta^{\rm Nm}_{\rm cdm} + 3H^{\rm Nm}_{\rm L}$$

$$[\partial_\tau + \mathcal{H}]\, \boldsymbol{v}^{\rm Nb}_{\rm cdm} = \nabla\Phi + \nabla\gamma^{\rm Nb} \qquad [\partial_\tau + \mathcal{H}]\, v^{\rm Nm}_{\rm cdm} = -k\Phi^{\rm N}$$

space threading $\boldsymbol{x} = \tilde{\boldsymbol{x}} + \boldsymbol{L}$ $\qquad H_{\rm T} = \tilde{H}_{\rm T} + kL$

$$\gamma^{\rm Nm} = \Phi^{\rm N} - \Phi$$

$$(\partial_\tau + \mathcal{H})\,\dot{H}_{\rm T} - 4\pi G a^2 \bar{\rho}_{\rm cdm}\,(H_{\rm T} - 3\zeta) = S$$

$$S = 4\pi G a^2\left(\bar{\rho}_{\rm other}\delta_{\rm other} + 3\mathcal{H}k^{-1}(\bar{\rho}+\bar{p})_{\rm other}(v - B) + 2\bar{p}\Pi\right)$$

Newtonian motion gauge spacetime

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Time slicing $B^{\rm Nm} = v^{\rm Nm}$ (not a unique choice)

$$g_{00} = -a^2\left(1 + 2A^{\rm Nm}\right) \,,$$
$$g_{0i} = a^2\, {\rm i}\hat{k}_i B^{\rm Nm} \,,$$
$$g_{ij} = a^2\left[\delta_{ij}\left(1 + 2H_{\rm L}^{\rm Nm}\right) + 2\left(\delta_{ij}/3 - \hat{k}_i\hat{k}_j\right)H_{\rm T}^{\rm Nm}\right]$$

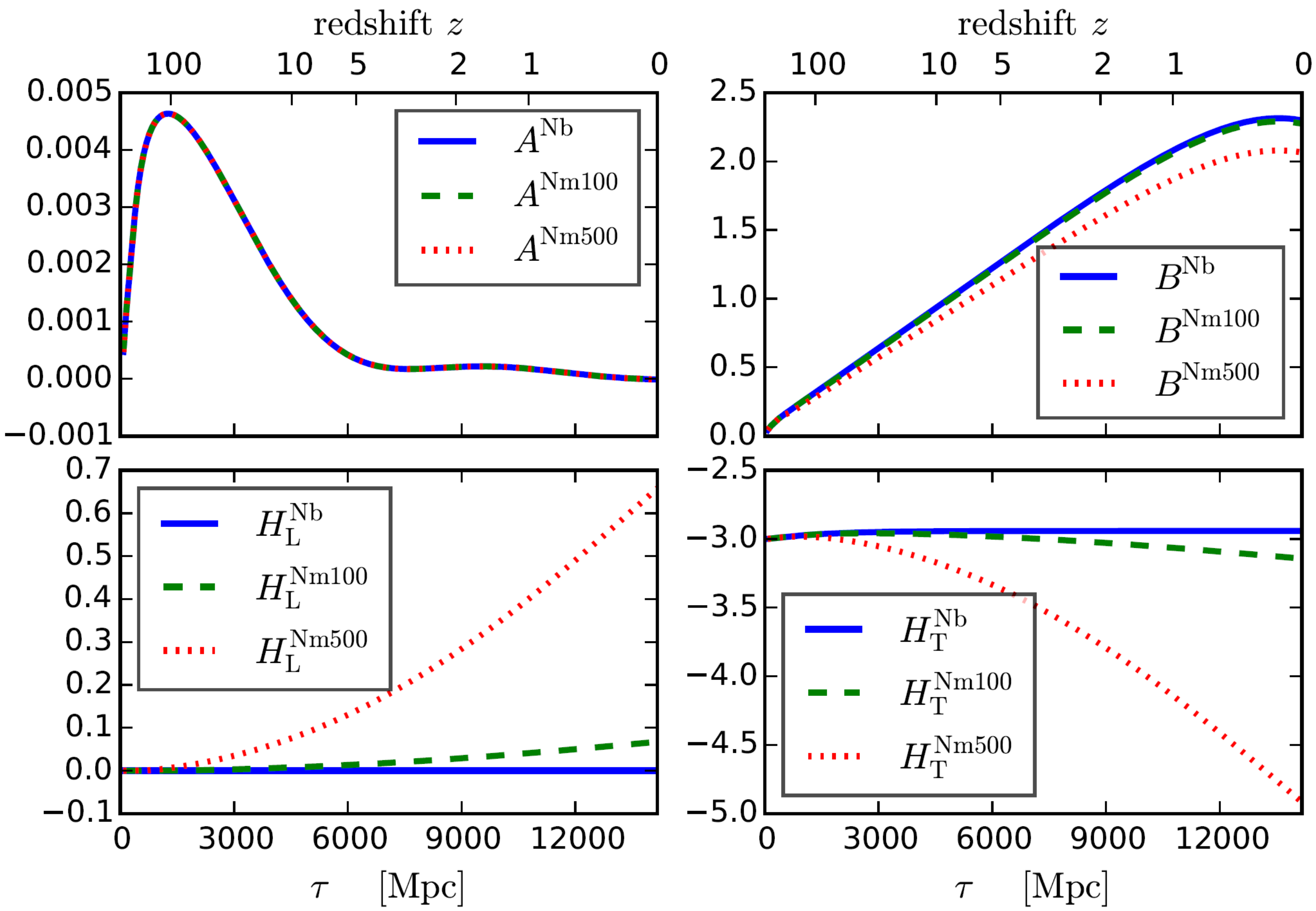
$$\left(\bar{\rho} + \bar{p}\right)A^{\rm Nm} = \frac{2}{3}\bar{p}\Pi - \delta p^{\rm Nm}$$

$$-4\pi G a^2 \bar{\rho}_{\rm cdm}(3H_{\rm L}^{\rm Nm} + \delta_{\rm cdm}^{\rm Nm}) = k^2 A^{\rm Nm} + \left(\partial_\tau + \mathcal{H}\right) k B^{\rm Nm}$$

$$\frac{1}{3}\dot{H}_{\rm T}^{\rm Nm} = \mathcal{H}A^{\rm Nm} - \dot{H}_{\rm L}^{\rm Nm}$$

Newtonian motion gauge metric

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221



$$\zeta = H_{\mathrm{L}} + \frac{1}{3} H_{\mathrm{T}} + \mathcal{H} k^{-1} (B - v)$$

$k = 10^{-3}\mathrm{Mpc}^{-1}$

$\zeta = -1$ on super-horizon

$H_{\mathrm{T}}^{\mathrm{Nb}} = 3\zeta = $ constant

$H_{\mathrm{L}}^{\mathrm{Nb}} = 0$

with no radiation

Application to N-body simulations

Fidler et.al. arXiv:1606.05588, arXiv:1702.03221

- Gauge transformation to N-body gauge

  At late times, radiation becomes negligible and N-body simulations are easier to interpret in N-body gauge

$$\boldsymbol{x}^{\mathrm{Nb}} = \boldsymbol{x}^{\mathrm{Nm}} + \boldsymbol{L}^{\mathrm{Nm}\to\mathrm{Nb}}, \qquad \boldsymbol{L}^{\mathrm{Nm}\to\mathrm{Nb}} = -k^{-1}\nabla L^{\mathrm{Nm}\to\mathrm{Nb}}$$

$$\ddot{L}^{\mathrm{Nm}\to\mathrm{Nb}} + \mathcal{H}\dot{L}^{\mathrm{Nm}\to\mathrm{Nb}} - 4\pi G a^2 \bar{\rho}_{\mathrm{cdm}} L^{\mathrm{Nm}\to\mathrm{Nb}} = -k\gamma^{\mathrm{Nb}} - 4\pi G a^2 k^{-1} \bar{\rho}_{\mathrm{other}} \delta^{\mathrm{Nb}}_{\mathrm{other}}$$

$$\delta^{\mathrm{Nb}}_{\mathrm{cdm}} - \delta^{\mathrm{N}} = -kL^{\mathrm{Nm}\to\mathrm{Nb}}$$

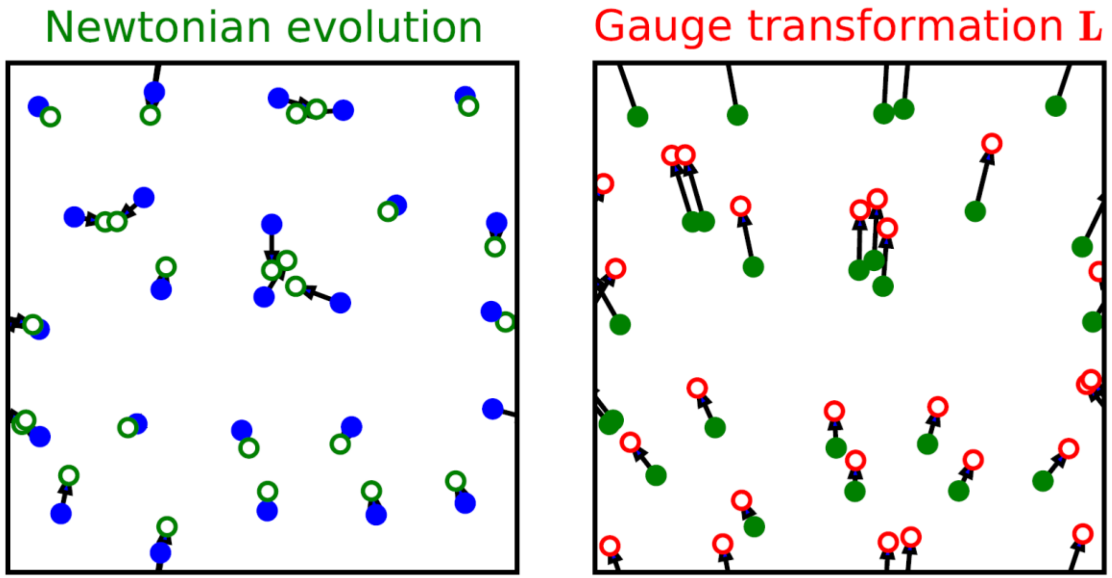
  this gauge transformation can be computed by linear Boltzmann code (CLASS)

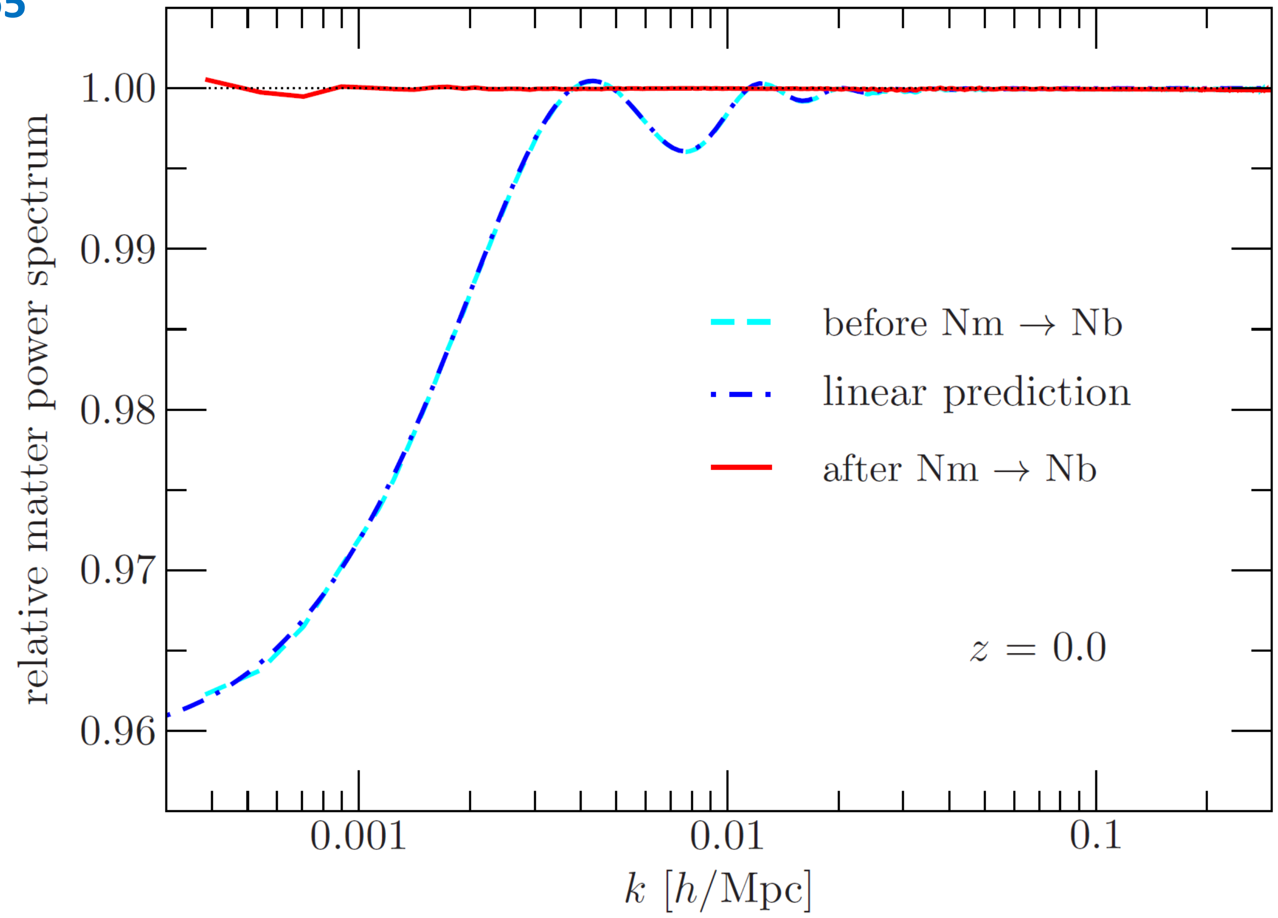
Comparison with relativistic simulations

Adamek et.al. arXiv:1703.08585

- gevolution Adamek et.al. arXiv:1604.06065

  Relativistic simulation code with weak field approximation





Weak field expansion

Fidler et.al. arXiv:1708.07769

- Non-linear density and velocity

  In Newtonian simulations, density and velocity become non-linear but the Newtonian potential remains small

$$\nabla^2\Phi(\boldsymbol{x},\tau) = -4\pi G\bar{\rho}a^2\delta(\boldsymbol{x},\tau)$$

- Weak field expansion

$$\nabla = k_{\rm ref}\mathcal{O}(\kappa) \qquad k_{\rm ref} = aH \equiv \mathcal{H} \qquad \Phi \text{ is of } \mathcal{O}(\epsilon)$$

  - Super/near horizon scales $\kappa^2 = \mathcal{O}(\epsilon)$ $\quad \delta = \mathcal{O}(\epsilon^2)$
  - Under horizon and linear scales $\kappa^2 = \mathcal{O}(1)$ (standard cosmological pert.)
  - Under horizon and "non-linear" scales $\kappa^2 = \mathcal{O}(\epsilon^{-1})$

$$\delta = \mathcal{O}(\kappa^2\epsilon) = \mathcal{O}(1) \qquad \kappa = \mathcal{O}(\epsilon^{-1/2})$$

Weak field expansion

Fidler et.al. arXiv:1708.07769

- Metric perturbations $(A,\, H_{\mathrm{L}},\, B,\, B_i,\, H_{\mathrm{T}},\, H_{\mathrm{T}i}$ and $H_{\mathrm{T}ij})$ are of order ϵ.

$$g_{00} = -a^2\left(1+2A\right)\,,$$

$$g_{0i} = -a^2\left(B_i + \hat{\nabla}_i B\right)\,,$$

$$\hat{\nabla}_i \equiv -(-\Delta)^{-1/2}\nabla_i \qquad \Delta = \nabla^2$$

$$g_{ij} = a^2\left[\delta_{ij}\left(1+2H_{\mathrm{L}}\right) + 2\left(\hat{\nabla}_i\hat{\nabla}_j + \frac{\delta_{ij}}{3}\right)H_{\mathrm{T}} - \hat{\nabla}_i H_{\mathrm{T}j} - \hat{\nabla}_j H_{\mathrm{T}i} - 2H_{\mathrm{T}ij}\right]$$

- Matter perturbations $\delta = \mathcal{O}(\kappa^2\epsilon) \qquad v = \mathcal{O}(\kappa\epsilon) \qquad \Sigma \sim \Sigma^i \sim \Sigma^{ij} = \mathcal{O}(\epsilon)$

$$T^0{}_0 = -\rho\,,$$

$$T^0{}_i = (\rho+p)(v_i + \hat{\nabla}_i v - B_i - \hat{\nabla}_i B)$$

$$T^i{}_j = p\delta^i_j + \left(\hat{\nabla}^i\hat{\nabla}_j + \frac{\delta^i_j}{3}\right)\Sigma - \frac{1}{2}\left(\hat{\nabla}^i\Sigma_j + \hat{\nabla}_j\Sigma^i\right) - \Sigma^i{}_j + (\rho+p)(v^i + \hat{\nabla}^i v)(v_j + \hat{\nabla}_j v)$$

Weak field Newtonian motion gauge

Fidler et.al. arXiv:1708.07769

- Temporal gauge

  $v = B$ violates our assumption

  we adopt $\mathfrak{K}B = \dot{H}_\mathrm{T}$ $\mathfrak{K} = (-\Delta)^{1/2}$

- Newtonian variables

$$4\pi G a^2 \delta\rho^{\mathrm{cdm}}_{\mathrm{count}} = \mathfrak{K}^2 \Phi^{\mathrm{N}} \qquad \delta^{\mathrm{cdm}}_{\mathrm{count}} = \delta + 3H_\mathrm{L} = \delta^{\mathrm{N}} \qquad v^{\mathrm{cdm}} = v^{\mathrm{N}}$$

- Newtonian motion gauge

$$A + (\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_\mathrm{T} = -\Phi^{\mathrm{N}}$$

This spatial gauge condition realises Newtonian (non-linear) Euler equation

$$(\partial_\tau + \mathcal{H})\left(v_i^{\mathrm{cdm}} + \hat{\nabla}_i v^{\mathrm{cdm}}\right) - (v^j_{\mathrm{cdm}} + \hat{\nabla}^j v^{\mathrm{cdm}})\hat{\nabla}_j \mathfrak{K}(v_i^{\mathrm{cdm}} + \hat{\nabla}_i v^{\mathrm{cdm}}) + \hat{\nabla}_i \mathfrak{K}\Phi^{\mathrm{N}}$$
$$= -\frac{1}{\rho}\left(\frac{2}{3}\hat{\nabla}_i \mathfrak{K}\Sigma^{\mathrm{cdm}} - \frac{1}{2}\mathfrak{K}\Sigma_i^{\mathrm{cdm}}\right)\,.$$

Linear Boltzmann + Newtonian N-body

Fidler et.al. arXiv:1708.07769

- Gauge conditions

$$(\partial_\tau + \mathcal{H})\dot{H}_{\mathrm{T}} = 4\pi G a^2 (\delta\rho_\gamma + 3\mathcal{H}(\rho_\gamma + p_\gamma)\mathfrak{K}^{-1}(v - \mathfrak{K}^{-1}\dot{H}_{\mathrm{T}}) - \rho_{\mathrm{cdm}}(3\zeta - H_{\mathrm{T}})) + 8\pi G a^2 \Sigma$$

$$\dot{H}_{\mathrm{T}} = \mathfrak{K} B$$

These are $\mathcal{O}(\epsilon)$ equations so can be computed using a linear Boltzmann code

Other variables can be computed using simulation quantities δ^{N} v^{N} Φ^{N}

$$A = -\Phi^{\mathrm{N}} - (\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}}\,,$$

$$H_{\mathrm{L}} = \Phi^{\mathrm{N}} - \frac{1}{3}H_{\mathrm{T}} - \gamma\,,$$

$$v = v^{\mathrm{N}}\,,$$

$$\delta = \delta^{\mathrm{N}} - 3H_{\mathrm{L}} = \delta^{\mathrm{N}} - 3\Phi^{\mathrm{N}} + H_{\mathrm{T}} + 3\gamma \qquad \gamma \equiv -(\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}} + 8\pi G a^2 \mathfrak{K}^{-2}\Sigma$$

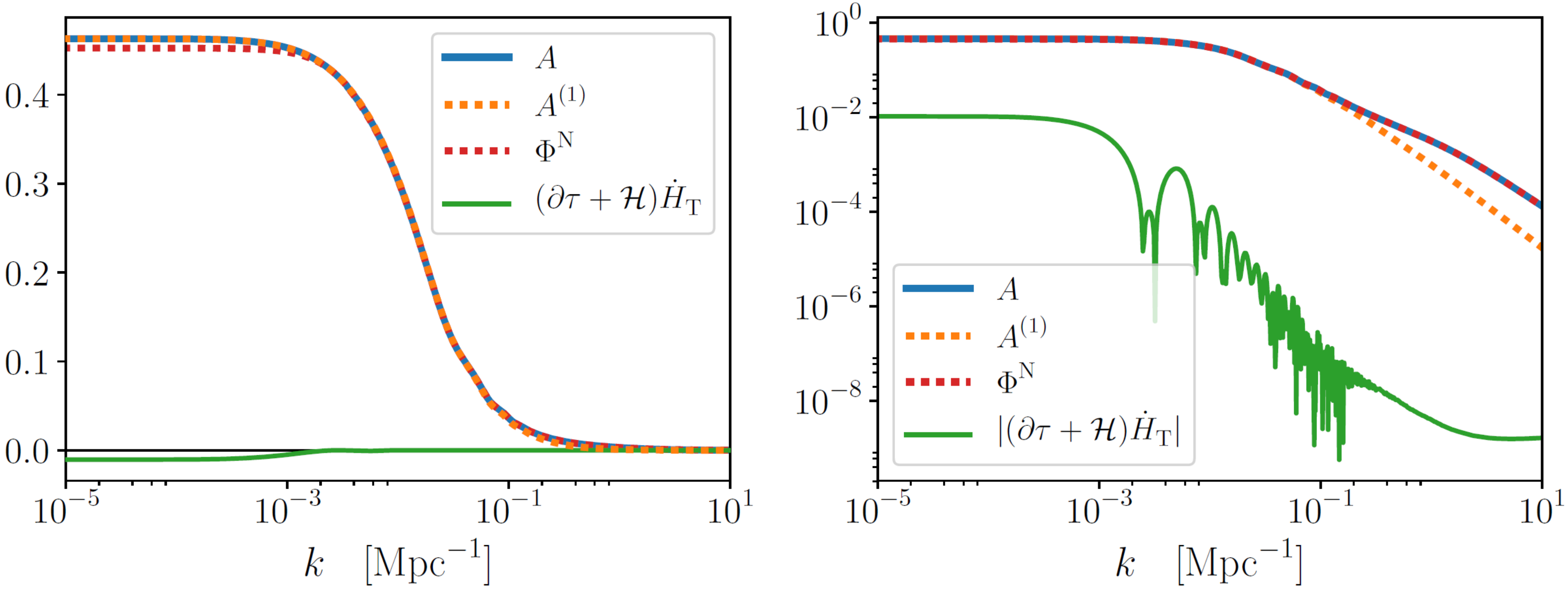
Linear Boltzmann + Newtonian N-body

Fidler et.al. arXiv:1708.07769

- Relativistic corrections (radiation) + non-linear corrections

$$A = -\boxed{\Phi^{\mathrm{N}}} - \boxed{(\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}}}$$

Relativistic correction



Poisson gauge

Fidler et.al. arXiv:1708.07769

- Relation to Poisson gauge (time slicing is the same)

$$g_{00} = -a^2(1+2\Psi)$$
$$g_{ij} = a^2\left[(1+2\Phi)\delta_{ij}\right]$$

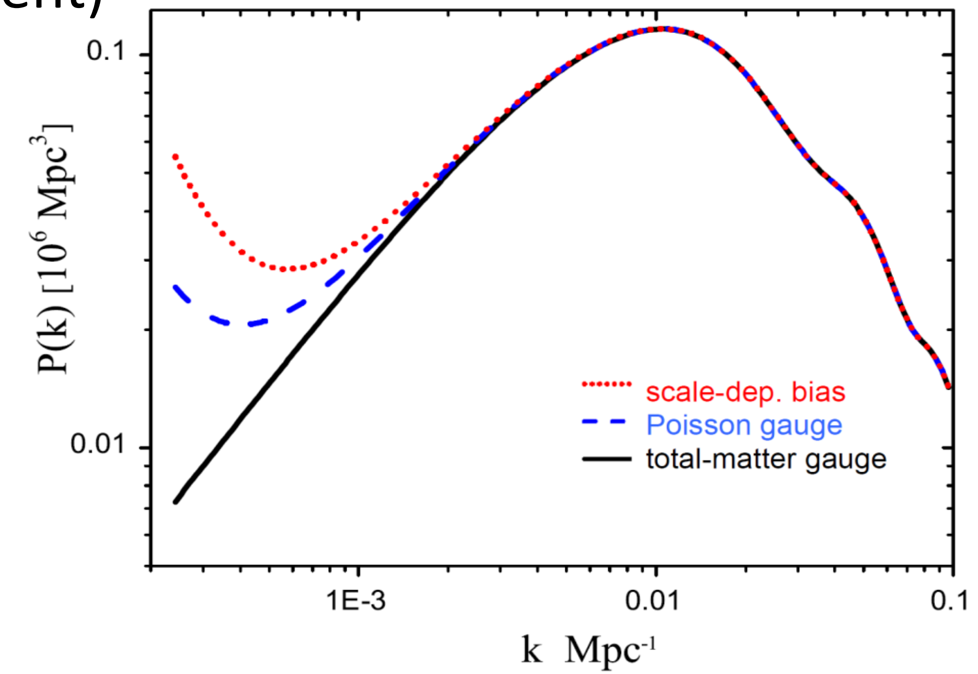
$$\Psi \equiv A^{\mathrm{P}} = A = -\Phi^{\mathrm{N}} - (\partial_\tau + \mathcal{H})\,\mathfrak{K}^{-2}\dot{H}_{\mathrm{T}}$$
$$\delta^{\mathrm{P}} = \delta = \delta^{\mathrm{N}} - 3\Phi^{\mathrm{N}} + H_{\mathrm{T}} + 3\gamma\,.$$
$$\Phi \equiv H_{\mathrm{L}}^{\mathrm{P}} = \Phi^{\mathrm{N}} - \gamma$$

(space threading is different)

$$v^{\mathrm{P}} = v^{\mathrm{N}} - \mathfrak{K}^{-1}\dot{H}_{\mathrm{T}}$$

- At late time (C.C. + CDM) $H_T = 3\zeta = \mathrm{const.}$

$$\delta^{\mathrm{P}} = \delta^{\mathrm{N}} - 3\Phi^{\mathrm{N}} + 3\zeta \qquad \Phi = \Phi^{\mathrm{N}}\,,$$
$$v^{\mathrm{P}} = v^{\mathrm{N}} \qquad \Psi = -\Phi = -\Phi^{\mathrm{N}}$$



Ray tracing

Fidler et.al. arXiv:1708.07769

- N-body results should be interpreted in Nm gauge

photon displacement

$$\delta x^0 = \left[-\delta a_0 + \Phi_0^{\mathrm{N}} + v_{||}^0\right]\chi - 2\int_0^{\chi_s} \mathrm{d}\chi \left[\Phi^{\mathrm{N}} + (\chi_s - \chi)\dot{\Phi}^{N}\right] + \int_0^{\tau_0} \mathrm{d}\tau \Phi^{\mathrm{N}}(0,\tau)$$

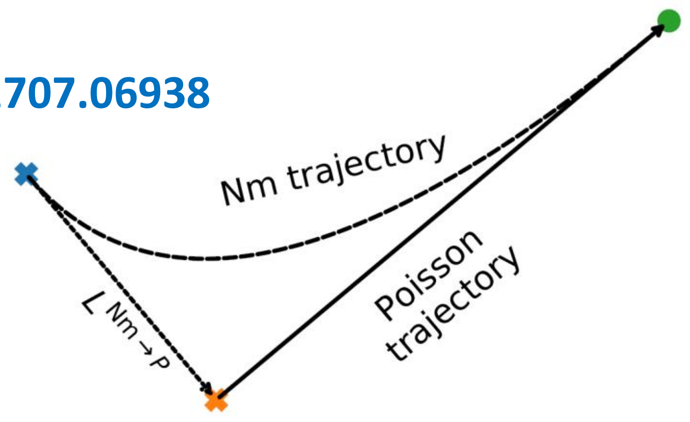
$$\delta x^i = \left[\delta a_0 n^i + \Phi_0^{\mathrm{N}} n^i - v_0^i - \hat{\nabla}^i v_0\right]\chi + 2\int_0^{\chi_s} \mathrm{d}\chi \left[-\Phi^{\mathrm{N}} n^i + (\chi_s - \chi)\nabla^i \Phi^{\mathrm{N}}\right] + \boxed{\hat{\nabla}^i \mathfrak{K}^{-1}(H_{\mathrm{T}e} - H_{\mathrm{T}0})}.$$

Integrated Coordinate Shift

see also Adamek arXiv:1707.06938

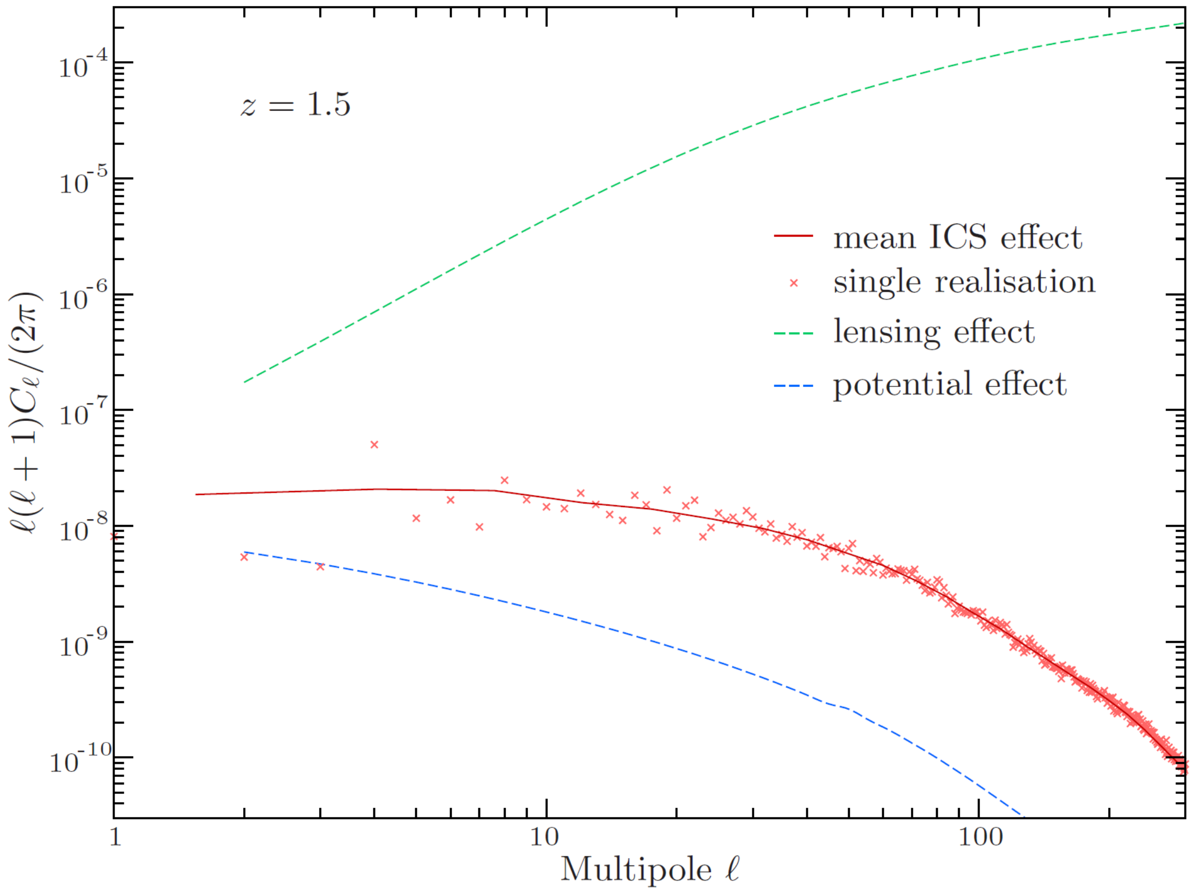
- Chisari & Zaldarriaga description

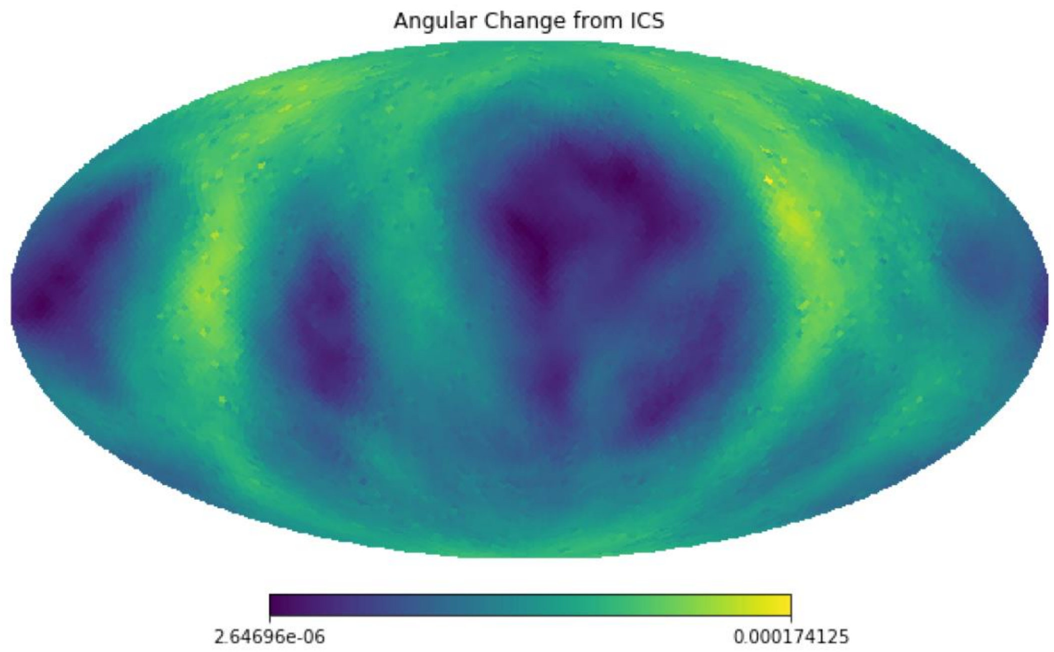
$$\boldsymbol{x}^{\mathrm{P}} = \boldsymbol{x}^{\mathrm{Nm}} + \delta\boldsymbol{x}_{\mathrm{in}}, \quad \nabla \cdot \delta\boldsymbol{x}_{\mathrm{in}} = -5\Phi_{\mathrm{in}}^{\mathrm{N}} = -\ 3\zeta$$



Chisari & Zaldarriaga arXiv:1101.3555

Integrated Coordinate Shift





Adamek & Fidler 1905.11721

Massive neutrinos, dark energy, modified gravity

- Massive neutrinos, dark energy, modified gravity

It is possible to include their effect *at linear level* **Tram et.al. 1811.00904, Dakin et.al. 1904.05210**

$$(\partial_\tau + \mathcal{H})\boldsymbol{v} = -\nabla\phi + \nabla\gamma \qquad \nabla^2\phi = 4\pi G a^2 \sum_\alpha \delta\rho_\alpha^{\rm Nb} \qquad \alpha \in \{\mathrm{cdm}, \mathrm{b}, \gamma, \nu, \mathrm{DE}\}$$

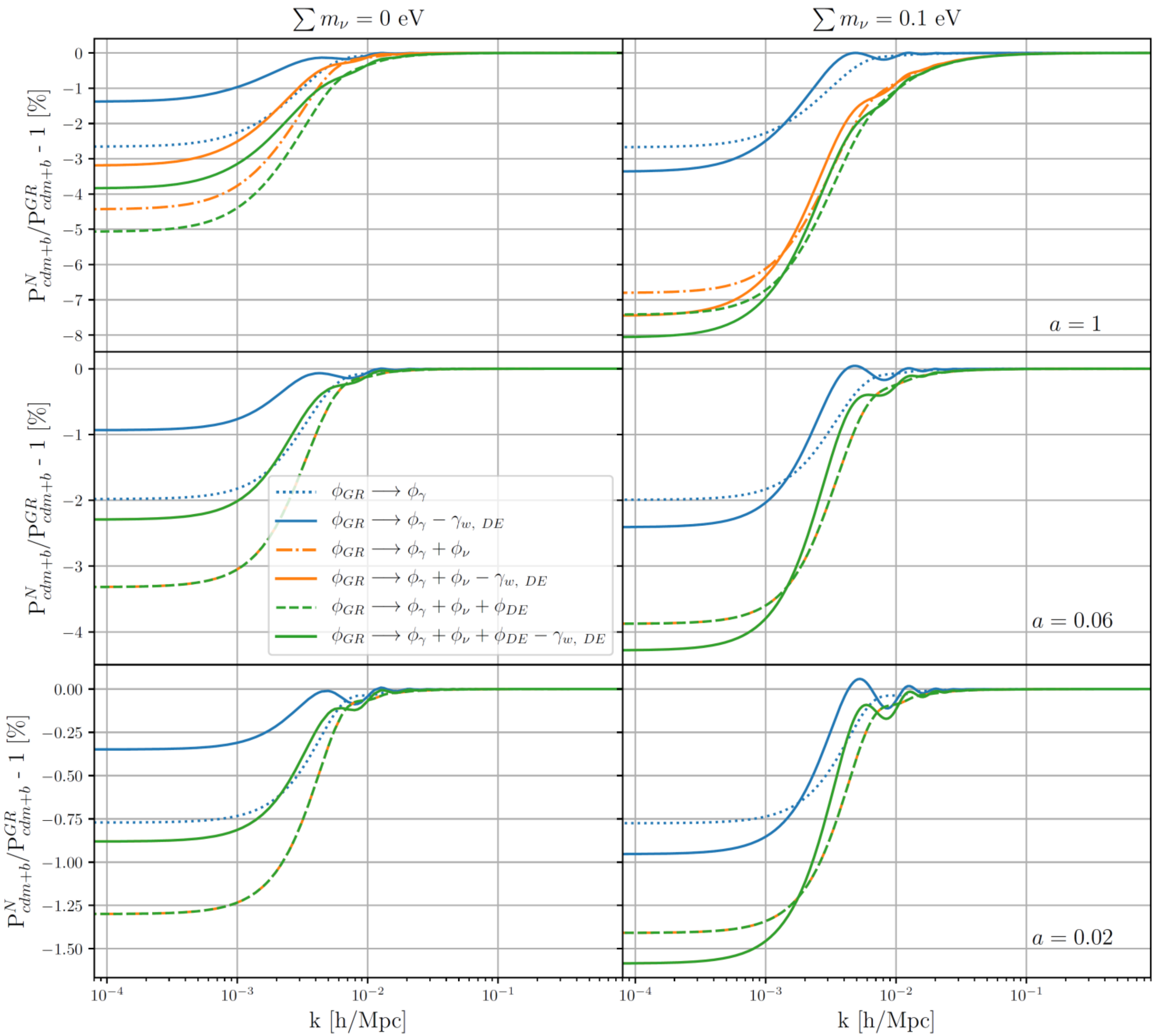
modified gravity **Brando, KK, Wands in preparation**

$$S_{\mu\nu}(g_{\mu\nu}) = 8\pi G T_{\mu\nu} \quad \Rightarrow \quad G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad T_{\mu\nu}^{\rm DE} \equiv \frac{1}{8\pi G}(G_{\mu\nu} - S_{\mu\nu})$$

$$\phi - \gamma^{\rm Nb} \equiv \phi_{\rm sim} + \phi_{\rm GR}$$

$$\nabla^2\phi_{\rm GR} \equiv \nabla^2(\phi_\gamma + \phi_\nu + \phi_{\rm DE} - \gamma^{\rm Nb})$$

this can be computed using linear Boltzmann code such as CAMB, CLASS, hi_CLASS, EFTCAMB



- K-essence

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + p(\phi, X)\right) + S_M$$

$$p(\phi, X) = \frac{V_0}{\phi^\alpha}\left(-X + X^2\right)$$

$$w_\phi = \frac{(1+w_B)\,\alpha}{2} - 1.$$

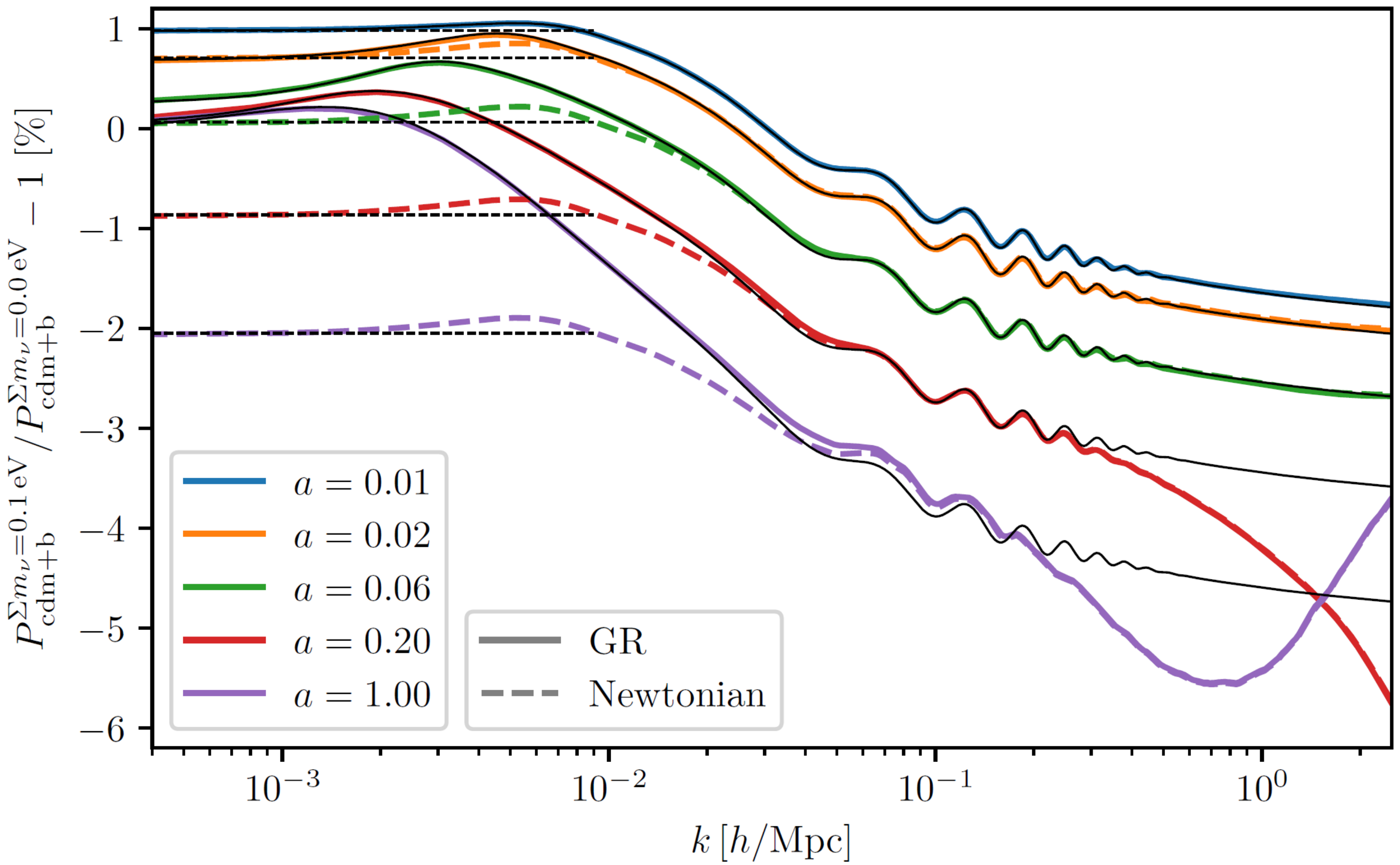
$$c_s^2 = \frac{\alpha(w_B+1)}{16 - 3\alpha(w_B+1)}$$

$$\alpha = 0.2, w_\phi = -0.9, c_s^2 = 0.013$$

Brando, KK, Wands in preparation

Massive neutrinos in simulations

Tram et.al. 1811.00904



Relative matter (CDM and baryons) power spectra between $\sum m_\nu = 0.1\,\mathrm{eV}$ and $\sum m_\nu = 0$

Newtonian:

νCONCEPT

(include non-linear neutrino densities)

GR:

PKGRAV+N body gauge

Questions and Answers

- Are Newtonian N-body simulations consistent with weak-field limit of GR?

  yes

- If so, how do we interpret Newtonian simulations in a relativistic framework?

  Newtonian N-body simulations should be interpreted in N-body gauge or Newtonian Motion gauge. Ray tracing needs to be done consistently in this gauge

- How do we include relativistic effects missing in simulations (e.g. radiation perturbations)

  Relativistic corrections can be included in spacetime metric perturbations in Newtonian motion gauge or as an additional force in N-body gauge using a linear Boltzmann code such as CAMB/CLASS

Further questions

- Backreaction problem

backreaction = the effect of inhomogeneities on the expansion

In Newtonian cosmology, backreaction is a boundary term thus it vanishes with a periodic boundary condition

Buchert gr-qc/9906015

- "Action-at-a-distance"

In Newtonian cosmology, the only gravitational equation is the Poisson equation, which is an elliptic equation

$$k^2\Phi^{\mathrm{N}} = 4\pi G a^2 \bar{\rho}_{\mathrm{cdm}} \delta^{\mathrm{N}}_{\mathrm{cdm}}$$

GR N-body

Shibata 1999 Prog. Theor. Phys. 101, 251 and 1199

- Geodesic

$$\frac{dx^i}{dt} = -\beta^i + \frac{\gamma^{ij}u_j}{u^0}$$

$$\frac{du_i}{dt} = -\alpha u^0 \alpha_{,i} + u_j \beta^j{}_{,i} - \frac{u_j u_k}{2u^0}\gamma^{jk}{}_{,i}$$

$$ds^2 = (-\alpha^2 + \beta_k\beta^k)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j$$

$$\gamma = \det(\gamma_{ij}) \equiv e^{12\phi},$$

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \text{ i.e., } \det(\tilde{\gamma}_{ij}) = 1$$

$$\tilde{A}_{ij} \equiv e^{-4\phi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right),$$

- Constraints

$$R - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 16\pi E,$$

$$D_i\tilde{A}^i{}_j - \frac{2}{3}D_j K = 8\pi J_j,$$

$$E = m_p \sum_{a=1}^{N} (u^0)_a \alpha e^{-6\phi}\delta^{(3)}(x^k - x_a^k),$$

$$J_i = m_p \sum_{a=1}^{N} (u_i)_a e^{-6\phi}\delta^{(3)}(x^k - x_a^k),$$

- Evolution equations (GR)

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\beta^k{}_{,j} + \tilde{\gamma}_{jk}\beta^k{}_{,i} - \frac{2}{3}\tilde{\gamma}_{ij}\beta^k{}_{,k} \quad \cdots\cdots$$

Weak field limit

- No hyperbolic equation for gravity

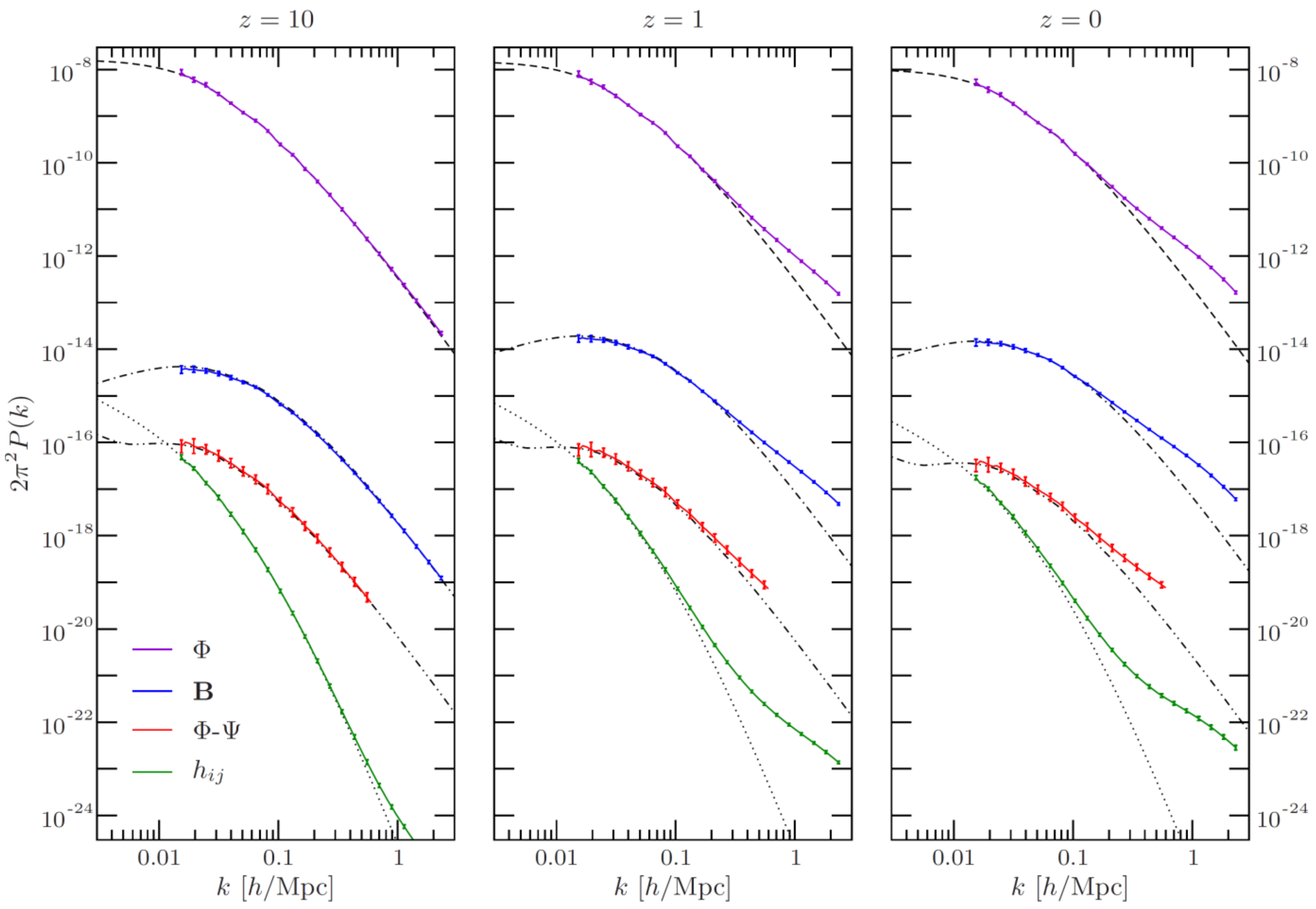
  The evolution equation for γ_{ij} decouples from matter evolution as it is suppressed compared with the Newton potential

- Relativistic corrections

  Relativistic corrections can be estimated from Newtonian simulations

  Bruni, Thomas & Wands 1306.1562

Adamek, Daveiro, Durrer & Kunz 1604.06065



GR simulations

See a recent code comparison paper Adamek et.al. 2003.08014

- Weak gravity N-body simulations

  gevolution: N-body simulations in Poisson gauge in weak gravity limit

  Adamek, Daveiro, Durrer & Kunz 1604.06065

- Fully constrained N-body simulations

  GRAMSES: N-body simulation with conformally flatness approximation

  Barrera-Hinojosa & Li arXiv:1905.08890

- Full GR simulations with dust

  Eloisa Bentivegna, Marco Bruni arXiv:1511.05124

  John T. Giblin, James B. Mertens, Glenn D. Starkman arXiv:1511.01105

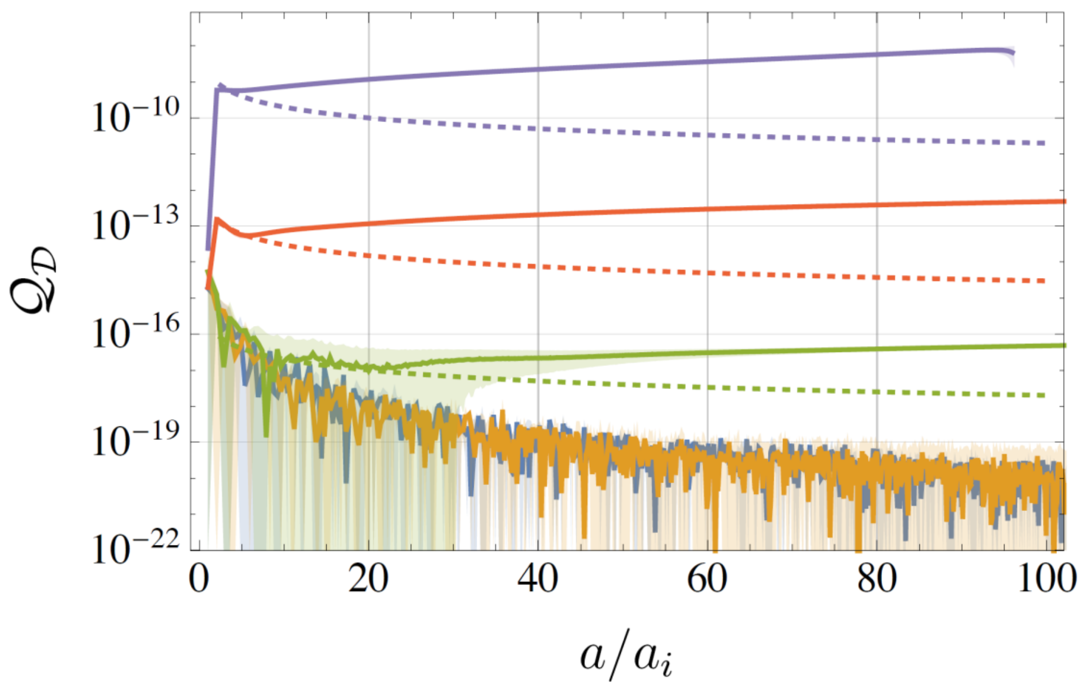
  Macperson, Price & Lasky arXiv:1807.01711

Backreaction

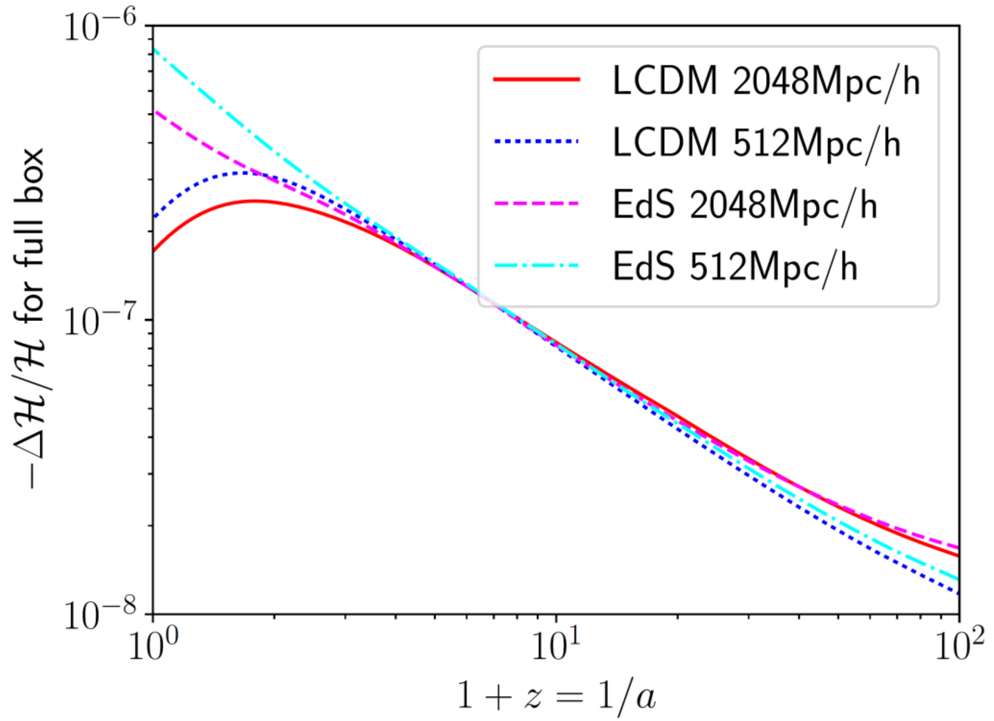
$$\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} = -\frac{4\pi}{3}\frac{M_{\mathcal{D}}}{a_{\mathcal{D}}^3} + \frac{\mathcal{Q}_{\mathcal{D}}}{3}$$

Buchert gr-qc/9906015

$$\tilde{\mathcal{H}} = \mathcal{H}\left(1 - 2\bar{\Phi} + \bar{\chi} - a\frac{d\bar{\Phi}}{da}\right) \qquad \chi = \Phi - \Psi$$



Bentivegna & Bruni arXiv: 1511.05124



Adamek et.al. arXiv:1707.06938

Conclusions

- Newtonian motion gauge

  We provided a framework to interpret and use Newtonian N-body simulations in terms of the weak filed limit of general relativity at leading order

  - inclusion of relativistic perturbations using a linear Einstein-Boltzmann code
  - identification of relativistic corrections to particle positions in N-body simulations
  - massive neutrinos/dark energy/modified gravity can be added using linear approximation

  Newtonian simulations can be used to construct light-cones consistently in GR

  This approach is being used in the Euclid flagship mock (v2)

Conclusions

- GR simulations

  there are still fundamental issues that cannot be addressed in Newtonian simulations

  - Backreaction
  - Propagating degrees of freedom in gravity – beyond “action-at-a-distance”
  - Relativistic corrections (these can be computed from Newtonian simulations)

  different approaches beyond the leading order

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